CSE541 Practice Midterm 2 Spring 2015
25 extra points

NAME **ID:**

All questions have REGULAR points assigned to them- as in regular test.

GRADE your test according to these points and write down the final GRADE you would get on the real test.

WRITE correct solutions for problems you didn't solve or got it wrong and bring it to class on TUESDAY

You will get up to **25 extra points** for the corrections - or for the test- in a case there was no need for corrections.

QUESTION 1 (15pts)

Let $S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \mathbf{A1}, \mathbf{A2}, \mathbf{A3},\ MP)$ be a proof system with the following axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The following Lemma holds in the system S.

LEMMA

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

B_1 $(A \Rightarrow B)$

B_2 $(\neg\neg A \Rightarrow A)$
Already PROVED

B_3 $(\neg\neg A \Rightarrow B)$

B_4 $(B \Rightarrow \neg\neg B)$

Already PROVED

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Already PROVED

B_7 $(\neg B \Rightarrow \neg A)$

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

QUESTION 2 (30pts)

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.

Consider any proof system S,

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ LA,\ (MP)\frac{A, (A \Rightarrow B)}{B})$$

that is **complete** under **classical** semantics.

Definition 1 Let $X \subseteq F$ be any subset of the set F of formulas of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of S.

We define a set $Cn(X)$ of all **consequences** of the set X as follows

$$Cn(X) = \{A \in F :\ X \vdash_S A\},$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(LA \cup X)$.

Part 1 (10pts)

(i) Prove that for any subsets X, Y of the set F of formulas the following **monotonicity property** holds.

If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

(ii) Prove that for any $X \subseteq F$, the set $\mathbf{T}$ of all propositional classical tautologies is a subset of $Cn(X)$, i.e.

$$\mathbf{T} \subseteq Cn(X)$$

Part 2 (20pts) Prove that for any $A, B \in F,\ X \subseteq F$,

$$(A \cap B) \in Cn(X) \ \ iff \ \ A \in Cn(X) \ and \ B \in Cn(X)$$

Hint: Use the Monotonicity and Completeness of S i.e. the fact that any tautology you might need for your proof is provable in S.

QUESTION 3 (20pts)

Let **GL** be the Gentzen style proof system for classical logic.

(1) Prove, by constructing a proper decomposition tree that

$\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

(2) Use the **completeness theorem** for **GL** to prove that

$\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

QUESTION 4 (20pts) Let **GL** be the Gentzen style proof system for classical logic.

1. Define SHORTLY Decomposition Tree for any A in **GL**.
item[2.] Prove Completeness Theorem for **GL**. We assume that the STRONG soundness has been proved.

QUESTION 5 (15pts)

We know that a classical tautology $(\neg(a \cap b) \cup (a \cap b))$ is NOT Intuitionistic tautology and we know by **Tarski Theorem** that $\neg\neg(\neg(a \cap b) \cup (a \cap b))$ is intuitionistically PROVABLE

FIND the proof of the formula

$$\neg\neg(\neg(a \cap b) \cup (a \cap b))$$

in the Gentzen system **LI** for Intuitionistic Logic.

1 GL Proof System

Axioms of GL

$$\Gamma'_1, a, \Gamma'_2 \;\longrightarrow\; \Delta'_1, a, \Delta'_2, \tag{1}$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$.

Inference rules of GL

The inference rules of **GL** are defined as follows.

Conjunction rules

$$(\cap \rightarrow)\ \frac{\Gamma^{'}, A, B, \Gamma \;\longrightarrow\; \Delta^{'}}{\Gamma^{'}, (A \cap B), \Gamma \;\longrightarrow\; \Delta^{'}}, \qquad (\rightarrow \cap)\ \frac{\Gamma \;\longrightarrow\; \Delta, A, \Delta^{'}\ ;\ \Gamma \;\longrightarrow\; \Delta, B, \Delta^{'}}{\Gamma \;\longrightarrow\; \Delta, (A \cap B), \Delta^{'}},$$

Disjunction rules

$$(\rightarrow \cup)\ \frac{\Gamma \;\longrightarrow\; \Delta, A, B, \Delta^{'}}{\Gamma \;\longrightarrow\; \Delta, (A \cup B), \Delta^{'}}, \qquad (\cup \rightarrow)\ \frac{\Gamma^{'}, A, \Gamma \;\longrightarrow\; \Delta^{'}\ ;\ \Gamma^{'}, B, \Gamma \;\longrightarrow\; \Delta^{'}}{\Gamma^{'}, (A \cup B), \Gamma \;\longrightarrow\; \Delta^{'}},$$

Implication rules

$$(\rightarrow\Rightarrow)\ \frac{\Gamma^{'}, A, \Gamma \;\longrightarrow\; \Delta, B, \Delta^{'}}{\Gamma^{'}, \Gamma \;\longrightarrow\; \Delta, (A \Rightarrow B), \Delta^{'}}, \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma^{'}, \Gamma \;\longrightarrow\; \Delta, A, \Delta^{'}\ ;\ \Gamma^{'}, B, \Gamma \;\longrightarrow\; \Delta, \Delta^{'}}{\Gamma^{'}, (A \Rightarrow B), \Gamma \;\longrightarrow\; \Delta, \Delta^{'}},$$

Negation rules

$$(\neg \rightarrow)\ \frac{\Gamma^{'}, \Gamma \;\longrightarrow\; \Delta, A, \Delta^{'}}{\Gamma^{'}, \neg A, \Gamma \;\longrightarrow\; \Delta, \Delta^{'}}, \qquad (\rightarrow \neg)\ \frac{\Gamma^{'}, A, \Gamma \;\longrightarrow\; \Delta, \Delta^{'}}{\Gamma^{'}, \Gamma \;\longrightarrow\; \Delta, \neg A, \Delta^{'}}.$$

2 LI Proof System

Axioms of LI

As the axioms of **LI** we adopt any sequent of the form

$$\Gamma_1, A, \Gamma_2 \;\longrightarrow\; A$$

for any formula $A \in \mathcal{F}$ and any sequences $\Gamma_1, \Gamma_2 \in \mathcal{F}^*$.

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules. They are defined as follows.

Structural Rules of LI

Weakening

$$(\rightarrow weak)\ \frac{\Gamma \;\longrightarrow}{\Gamma \;\longrightarrow\; A}\ .$$

A is called the weakening formula.

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta},$$

A is called the contraction formula , Δ contains at most one formula.

Exchange

$$(exchange \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \longrightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \longrightarrow \Delta},$$

Δ contains at most one formula.

Logical Rules of LI

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta}, \qquad (\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)},$$

Δ contains at most one formula.

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}, \qquad (\rightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \; ; \; B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}, \qquad (\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \; ; \; B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow}, \qquad (\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}.$$