cse541
LOGIC FOR COMPUTER SCIENCE

Professor Anita Wasilewska

Spring 2015

LECTURE 5a

Short REVIEW Chapters1 -5

DEFINITIONS: Chapter 3 and Chapter 4

Here I repeat for you some basic **DEFINITIONS** from Chapters 3, 4 and 5

You have to prepare them for MIDTERM 1

I will ask you to **WRITE** down a full, correct text of 1-3 of them - in EXACTLY the same form as they are presented here

Knowing all basic **Definitions** is the first step to understanding the material

DEFINITIONS

Definition 1

A **propositional language** is a pair

$$\mathcal{L} = (\mathcal{A}, \mathcal{F})$$

where $\mathcal{A}, \mathcal{F}$ are called the **alphabet** and a set of **formulas**, respectively

Definition 2

An **alphabet** is a set $\mathcal{A} = VAR \cup CON \cup PAR$

VAR, CON, PAR are all disjoint sets of propositional variables, connectives and parenthesis, respectively

We assume that

1. $PAR = \{(,\ 0)\}$

2. VAR is a countably infinite set and denote elements of VAR by $a, b, c, d, ...,$ with indices if necessary

DEFINITIONS

2. $CON \neq \emptyset$ **contains only** unary and binary connectives, i.e. $CON = C_1 \cup C_2$ where

C_1 is the set of all unary connectives, and

C_2 is the set of all binary connectives

Language Notation

We denote the language $\mathcal{L}$ with the set of connectives CON by

$$\mathcal{L}_{CON}$$

Metalanguage notation; we use the **set union** symbol $\cup$ when needed; it is clear from the context that it is not the connective $\cup$ symbol from our language $\mathcal{L}$

DEFINITIONS

Definition 3

The set $\mathcal{F}$ of **all formulas** of a propositional language $\mathcal{L}_{CON}$ is build **recursively** from the elements of the alphabet $\mathcal{A}$ as follows

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the **smallest** set for which the following conditions are satisfied

(1) $VAR \subseteq \mathcal{F}$

(2) If $A \in \mathcal{F}$, $\triangledown \in C_1$, then $\triangledown A \in \mathcal{F}$

(3) If $A, B \in \mathcal{F}$, $\circ \in C_2$ i.e $\circ$ is a two argument connective, then

$(A \circ B) \in \mathcal{F}$

Propositional variables are formulas and they are called **atomic formulas**

Question Example

Question

Use **Definitions 1,2, 3** to **define** the language $\mathcal{L}_{\{\neg,\ K,\ \cap,\ \Rightarrow\}}$ where K is one argument knowledge connective

Solution

$$\mathcal{L}_{\{\neg, K, \cap,\ \Rightarrow\}} = (\mathcal{A}, \mathcal{F})$$

The components $\mathcal{A},\ \mathcal{F}$ are defined as follows

Alphabet is

$$\mathcal{A} = VAR \cup \{\neg,\ K,\ \cap,\ \Rightarrow\} \cup \{(,)\}$$

Question Example

The set of all **formulas** is defined as follows

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the **smallest** set for which the following conditions are satisfied

(1) $VAR \subseteq \mathcal{F}$
(2) If $A \in \mathcal{F}$, then $\neg A,\ KA \in \mathcal{F}$
(3) If $A, B \in \mathcal{F}$, then
$(A \cap B),\ (A \Rightarrow B) \in \mathcal{F}$

DEFINITIONS: Extension

Definition 4

Given the truth assignment (in classical semantics)

$v : VAR \longrightarrow \{T, F\}$

We define its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as $v^* : \mathcal{F} \longrightarrow \{T, F\}$ such that

(i) for any $a \in VAR$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$

DEFINITIONS: Satisfaction Relation

Definition 5 Let $v : VAR \longrightarrow \{T, F\}$

We say that

v **satisfies** a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

Definition: We say that

v **does not satisfy** a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$

DEFINITIONS: Model, Counter-Model, Tautology

Definition 6

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow \{T, F\}$

We say that

v is a **model** for A iff $v \models A$

v is a **counter-model** for A iff $v \not\models A$

Definition 7

A is a **tautology** iff for any $v : VAR \longrightarrow \{T, F\}$ we have that $v \models A$

Notation

We write symbolically $\models A$

DEFINITIONS: Restricted Truth Assignments

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

Definition 8 Given a formula $A \in \mathcal{F}$, any function

$$v_A : \quad VAR_A \longrightarrow \{T, F\}$$

is called a truth assignment restricted to A

DEFINITIONS: Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ and let $\mathcal{S} \neq \emptyset$ be any non empty set of formulas of $\mathcal{L}$, i.e.

$$\mathcal{S} \subseteq \mathcal{F}$$

Definition 9

A truth truth assignment $v : VAR \longrightarrow \{T, F\}$

is a **model for the set** $\mathcal{S}$ of formulas if and only if

$$v \models A \quad \text{for all formulas} \quad A \in \mathcal{S}$$

We write

$$v \models \mathcal{S}$$

to denote that v **is a model for the set** $\mathcal{S}$ of formulas

DEFINITIONS: Consistent Sets of Formulas

Definition 10

A set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** v such that $v \models \mathcal{G}$

Otherwise $\mathcal{G}$ is called **inconsistent**

DEFINITIONS: Independent Statements

Definition 11

A formula A is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$

if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent**

if and only if

$$\mathcal{G} \cup \{A\} \quad \text{and} \quad \mathcal{G} \cup \{\neg A\} \quad \text{are } \textbf{consistent}$$

DEFINITIONS: Many Valued Extensional Semantics **M**

The extensional semantics **M** is defined for a non-empty set of V of **logical values of any cardinality**

We only **assume** that the set V of logical values of **M** always has a special, distinguished logical value which serves to define a notion of tautology

We denote this distinguished value as T

Formal definition of **many valued extensional semantics M** for the language $\mathcal{L}_{CON}$ consists of giving **definitions** of the following main components:

1. Logical Connectives under semantics **M**
2. Truth Assignment for **M**
3. Satisfaction Relation, Model, Counter-Model under semantics **M**
4. Tautology under semantics **M**

Definition of **M** - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Let V be a non-empty set of **logical values** adopted by the semantics **M**

Definition

Connectives $\triangledown \in C_1$, $\circ \in C_2$ are called **M -extensional** iff their semantics **M** is defined by respective functions

$$\triangledown : \; V \longrightarrow V \quad \text{and} \quad \circ : \; V \times V \longrightarrow V$$

DEFINITION: Definability of Connectives under a semantics **M**

Given a propositional language $\mathcal{L}_{CON}$ and its **extensional semantics M**

We adopt the following definition

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ **under the semantics M** if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are **defined by the semantics M**

DEFINITION: **M** Truth Assignment Extension v^* to $\mathcal{F}$

Definition

Given the **M** truth assignment $v : VAR \longrightarrow V$

We define its **M extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function $v^* : \mathcal{F} \longrightarrow V$, such that the following conditions are satisfied

(i) for any $a \in VAR$

$$v^*(a) = v(a);$$

(ii) For any connectives $\triangledown \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\triangledown A) = \triangledown v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

DEFINITION: **M** Satisfaction, Model, Counter Model, Tautology

Definition: Let $v : VAR \longrightarrow V$

Let $T \in V$ be the **distinguished logical value**

We say that
v **M** satisfies a formula $A \in \mathcal{F}$ ($v \models_{\mathbf{M}} A$) iff
$v^*(A) = T$

Definition:

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow V$

Any v such that $v \models_{\mathbf{M}} A$ is called a **M model** for A

Any v such that $v \not\models_{\mathbf{M}} A$ is called a **M counter model** for A

A is a **M tautology** ($\models_{\mathbf{M}} A$) iff $v \models_{\mathbf{M}} A$, **for all**
$v : VAR \longrightarrow V$

Chapter 5: Challenge Exercise

1. Define your own propositional language $\mathcal{L}_{CON}$ that contains also **different connectives** that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language $\mathcal{L}_{CON}$ **does not need** to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for **your own semantic**

4. Define formally your own **extensional semantics M** for your language $\mathcal{L}_{CON}$ - it means

write carefully all **Steps 1- 4** of the definition of your **M**

Challenge Problems

Work on **Challenge Problems** posted in Lectures 3-5

Chapter 3: Question 1

Question 1 Write the following natural language statement:

From the fact that it is not necessary that a red flower is not a bird we deduce that:

it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies

as a formulas of two languages

1. $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$
2. $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Chapter 3: Question 1

Solution The statement is :

From the fact that it is not necessary that a red flower is not a bird we deduce that:

it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies

1. We translate our statement into a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b , where

a denotes statement: *red flower is a bird* ,

b denotes statement: *a bird flies*

Propositional Modal Connectives: C, I

C denotes statement: it is possible that, **I** denotes statement: it is necessary that

Chapter 3: Question 1

Solution The statement is :

From the fact that it is ***not*** *necessary that a red flower is* ***not*** *a bird we* ***deduce*** *that:*

it is ***not*** *possible that the red flower is a bird* ***or****, if it is possible that the red flower is a bird, then it is* ***not*** *necessary that a bird flies*

Translation for the language $\mathcal{L}_{\{\neg,\mathbf{C},\mathbf{I},\cap,\cup,\Rightarrow\}}$ is

$$A_1 = (\neg\mathbf{I}\neg a \Rightarrow (\neg\mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg\mathbf{I}b)))$$

Observe that you could also use symbols $\Box$ for necessity and $\Diamond$ for possibility but in this case the formula would belong to the language $\mathcal{L}_{\{\neg,\Diamond,\ \Box,\ \cap,\ \cup,\ \Rightarrow\}}$ and hence **not to the language** $\mathcal{L}_{\{\neg,\ \mathbf{C},\ \mathbf{I},\ \cap,\ \cup,\ \Rightarrow\}}$ as stated in the **Question**

Chapter 3: Question 1

The statement is :

From the fact that it is not necessary that a red flower is not a bird we deduce that:

it is not possible that the red flower is a bird or, if it is possible that the red flower is a bird, then it is not necessary that a bird flies

2. Now we **translate** our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ as follows

Propositional Variables: a, b, c

a denotes statement: *it is necessary that a red flower is not a bird*

b denotes statement: *it is possible that a red flower is a bird*

c denotes a statement: *it is necessary that a bird flies*

Translation

$$A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

Chapter 3: Question 2

Question 2

1. Determine the main connectives and degrees of the formulas from **Q4**, i.e. of

$$A_1 = (\neg\mathbf{I}\neg a \Rightarrow (\neg\mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg\mathbf{I}b))),$$

$$A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

Solution

Main connective of the formula A_1 is: $\Rightarrow$

Main connective of the formula A_2 is also $\Rightarrow$

Degree of the formula A_1 is: 11

Degree of the formula A_2 is: 6

Chapter 3: Question 2

2. Determine all proper, non-atomic sub-formulas of A_1, and non-atomic sub-formulas of A_2

Solution

All proper, non-atomic sub-formulas of A_1 are

$$\neg\mathbf{I}\neg a, (\neg\mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg\mathbf{I}b)), \mathbf{I}\neg a, \neg a, \neg\mathbf{C}a, (\mathbf{C}a \Rightarrow \neg\mathbf{I}b), \mathbf{C}a, \neg\mathbf{I}b, \mathbf{I}b$$

All non-atomic sub-formulas of A_2 are:

$$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)), \neg a, (\neg b \cup (b \Rightarrow \neg c)), \neg b, (b \Rightarrow \neg c), \neg c$$

CHAPTER 4: Question 3

Question 3

1. Find a restricted model for formula A, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can't use** short-hand notation

Show each step of solution

Solution

For any formula A, we denote by VAR_A a set of all variables that appear in A

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \longrightarrow \{T, F\}$ is called a truth assignment restricted to A

Chapter 4: Question 3

Let $v : VAR \longrightarrow \{T, F\}$ be any truth assignment such that

$$v(a) = v_A(a) = T,\ v(b) = v_A(b) = T,\ v(c) = v_A(c) = F$$

We evaluate the value of the **extension** v^* of v on the formula A as follows

$v^*(A) = v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))$

$= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c)))$

$= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c)))$

$= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c)\ ))$

$= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)\ ))$

$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$$v_A \models A \quad \text{and} \quad v \models A$$

Question 4

1. Find a restricted model and a restricted counter-model for A, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You **can use** short-hand notation. Show work

Solution

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : \ VAR_A \longrightarrow \{T, F\}$ is called a truth assignment restricted to A

We define now $v_A(a) = T, \ v_A(b) = T, \ v_A(c) = F$, in shorthand: $a = T, b = T, c = F$ and evaluate $(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$$v_A \models A$$

Chapter 4: Question 4

Observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = T$ when $a = T$ and b, c any truth values as by definition of implication we have that $F \Rightarrow \text{anything} = T$

Hence $a = T$ gives us 4 models as we have 2^2 possible values on b and c

Chapter 4: Question 4

We take as a restricted counter-model: a=F, b =T and c=T

Evaluation: observe that

$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if

$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if

$a = F,\ \neg b = F$ and $(b \Rightarrow \neg c) = F$ if and only if

$a = F, b = T$ and $(T \Rightarrow \neg c) = F$ if and only if

$a = F, b = T$ and $\neg c = F$ if and only if

$a = F, b = T$ and $c = T$

The above proves also that a=F, b =T and c=T is the only **restricted counter -model** for A

Chapter 4: Question 5

Question 5 Justify whether the following statements **true** or **false**

S1 There are more then 3 possible restricted counter-models for A

S2 There are more then 2 possible restricted models of A

Solution

Statement: There are more then 3 possible restricted counter-models for A is **false**

We have just proved that there is only one possible restricted counter-model for A

Statement: There are more then 2 possible restricted models of A is **true**

There are 7 possible restricted models for A

Justification: $2^3 - 1 = 7$

Chapter 4: Question 6

Question 6

1. List 3 **models** and 2 **counter-models** for A from **Question 3**, i.e.

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are **extensions** to the set VAR of all variables of **one** the restricted models and of **one** of the restricted counter-models that you have found in **Questions 3, 4**

Chapter 4: Question 6

Solution

One of the **restricted models** is, for example a function

$v_A : \{a, b, c\} \longrightarrow \{T, F\}$ such that

$v_A(a) = T,\ v_A(b) = T,\ v_A(c) = F$

We **extend** v_A to the set of all propositional variables VAR to obtain a (non restricted) **models** as follows

Chapter 4: Question 6

Model w_1 is a function

$w_1 : VAR \longrightarrow \{T, F\}$ such that

$w_1(a) = v_A(a) = T$, $w_1(b) = v_A(b) = T$,

$w_1(c) = v_A(c) = F$, and $w_1(x) = T$, for all $x \in VAR - \{a, b, c\}$

Model w_2 is defined by a formula

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,

$w_2(c) = v_A(c) = F$, and $w_2(x) = F$, for all $x \in VAR - \{a, b, c\}$

Chapter 4: Question 6

Model w_3 is defined by a formula

$w_3(a) = v_A(a) = T,\ w_3(b) = v_A(b) = T,\ w_3(c) = v(c) = F,$

$w_3(d) = F$ and $w_3(x) = T$ for all $x \in VAR - \{a, b, c, d\}$

There is as many of such models, as extensions of v_A to the set VAR, i.e. as many as real numbers

Chapter 4: Question 6

A counter-model for a formula A, by **definition**, is any function

$$v : VAR \longrightarrow \{T, F\}$$

such that $v^*(A) = F$

A restricted counter-model for A (only one as proved in **question 5**) is a function

$$v_A : \{a, b\} \longrightarrow \{T, F\}$$

such that such that

$$v_A(a) = F, \;\; v_A(b) = T, \;\; v_A(c) = T$$

Chapter 4: Question 6

We extend v_A to the set of all propositional variables VAR to obtain (non restricted) some counter-models.

Here are **two** of such extensions

Counter- model w_1:

$w_1(a) = v_A(a) = F$, $w_1(b) = v_A(b) = T$,
$w_1(c) = v(c) = T$, and $w_1(x) = F$, for all
$x \in VAR - \{a, b, c\}$

Counter- model w_2:

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,
$w_2(c) = v(c) = T$, and $w_2(x) = T$ for all
$x \in VAR - \{a, b, c\}$

There is as many of such **counter- models**, as extensions of v_A to the set VAR, i.e. as many as real numbers

Chapter 4: Models for Sets of Formulas

Definition

A truth assignment v is a **model for a set** $\mathcal{G} \subseteq \mathcal{F}$ **of formulas** of a given language $\mathcal{L} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$

if and only if

$$v \models B \quad \textbf{for all} \quad B \in \mathcal{G}$$

We denote it by $v \models \mathcal{G}$

Observe that the set $\mathcal{G} \subseteq \mathcal{F}$ can be **finite** or **infinite**

Chapter 4: Consistent Sets of Formulas

Definition

A set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** v such that $v \models \mathcal{G}$

Otherwise $\mathcal{G}$ is called **inconsistent**

Chapter 4: Independent Statements

Definition

A formula A is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$

if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent**

if and only if

$$\mathcal{G} \cup \{A\} \quad \text{and} \quad \mathcal{G} \cup \{\neg A\} \quad \text{are } \textbf{consistent}$$

Chapter 4: Question 7

Question 7

Given a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Show that $\mathcal{G}$ is **consistent**

Solution

We have to find $v : VAR \longrightarrow \{T, F\}$ such that

$$v \models \mathcal{G}$$

It means that we need to bf find v such that

$$v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T$$

Chapter 4: Question 7

Observe that $\models ((a \cap b) \Rightarrow b)$, hence we have that

1. $v^*((a \cap b) \Rightarrow b) = T$ for any v

$v^*(\neg a) = \neg v^*(a) = \neg v(a) = T$ **only** when $v(a) = F$ hence

2. $v(a) = F$

$v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T$
only when $v(b) = T$ so we get

3. $v(b) = T$

This **means** that for any $v : VAR \longrightarrow \{T, F\}$ such that
$v(a) = F, \;\; v(b) = T$

$$v \models \mathcal{G}$$

and we **proved** that $\mathcal{G}$ is **consistent**

Chapter 4: Question 8

Question 8

Show that a formula $A = (\neg a \cap b)$ is **not independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Solution

We have to show that **it is impossible** to construct v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

Observe that we have just proved that any v such that $v(a) = F$, and $v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ for $\mathcal{G}$ so we have to check now if it is **possible** that $v \models A$ and $v \models \neg A$

Chapter 4: Question 8

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for

$v(a) = F,$ and $v(b) = T$

$v^*(A) = v^*((\neg a \cap b) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This end the proof that A is **not independent** of $\mathcal{G}$

Chapter 4: Question 9

Question 9

2. Find an infinite number of formulas that are **independent** of $\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$

This **my solution** - there are many others- this one seemed to me the **most simple**

Solution

We just proved that any v such that $v(a) = F,\ \ v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ and so all other possible models for $\mathcal{G}$ must be **extensions** of v

Chapter 4: Question 9

We **define** a countably infinite set of formulas (and their negations) and corresponding **extensions** of v (restricted to to the set of variables $\{a, b\}$) such that $v \models \mathcal{G}$ as follows

Observe that **all extensions** of v restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$VAR - \{a,\ b\} = \{a_1,\ a_2,\ \ldots, a_n \ldots\}$$

We take as a set of formulas (to be **proved to be independent**) the set of **atomic formulas**

$$\mathcal{F}_0 = \{a_1,\ a_2,\ \ldots, a_n \ldots\}$$

Chapter 4: Question 9

We define now two sequences

$\{v_i\}_{n\geq 1}$ and $\{w_i\}_{n\geq 1}$ of **extensions** of v as follows

$v_i : VAR - \{a,\ b\} \longrightarrow \{T, F\}$ is such that $v_i(a_i) = T$

$w_i : VAR - \{a,\ b\} \longrightarrow \{T, F\}$ is such that $w_i(a_i) = F$

By definition of the **extension** we have that

$v_i \models \mathcal{G}$ $w_i \models \mathcal{G}$ for all $i \geq 1$ and

$$v_i \models \mathcal{G} \cup \{a_i\} \quad \text{and} \quad w_i \models \mathcal{G} \cup \{\neg a_i\}$$

This **proves** that each formula $a_i \in \mathcal{F}_0$ is **independent** of the set $\mathcal{G}$

CHAPTER 5
Some Extensional Many Valued Semantics

Chapter 5: Question 10

Question 10

We **define** a 4 valued $\mathbf{H_4}$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $\mathbf{H_4}$ are operations in the set $\{F, \perp_1, \perp_2, T\}$, where $\{F < \perp_1 < \perp_2 < T\}$ and are defined as follows

Conjunction $\cap$ is a function

$\cap : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, , T\}$,

such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$

$$a \cap b = min\{a, b\}$$

Chapter 5: Many Valued Semantics

Disjunction $\cup$ is a function

$\cup: \ \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$

$$a \cup b = max\{a, b\}$$

Implication $\Rightarrow$ is a function

$\Rightarrow: \ \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$,

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

Negation:

$$\neg a = a \Rightarrow F$$

Chapter 5: Question 10

Part 1 Write Truth Tables for IMPLICATION and NEGATION in **$\mathbf{H}_4$**

Solution

H_4 Implication

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	F	T	T	T
$\bot_2$	F	$\bot_1$	T	T
T	F	$\bot_1$	$\bot_2$	T

H_4 Negation

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	F	F	F

Chapter 5: Question 10

Part 2 Verify whether

$$\models_{\mathbf{H}_4}((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution

Take any v such that

$v(a) = \bot_1 \quad v(b) = \bot_2$

Evaluate

$v * ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) =$
$T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2$

This proves that our v is a **counter-model** and hence

$$\not\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Chapter 6: Classical Propositional Tautologies

Question 11

Show that (can't use TTables!)

$$\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$$

Solution

Denote $A = (\neg a \cup b)$, and $B = ((c \cap d) \Rightarrow \neg d)$

Our formula becomes a substitution of a **basic tautology**

$$(A \Rightarrow (B \Rightarrow A))$$

and hence is a **tautology**