cse541
LOGIC FOR COMPUTER SCIENCE

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LECTURE 12

Chapter 12
Gentzen Sequent Calculus **LI** for Intuitionistic Logic

Original Gentzen System **LI** for Intuitionistic Logic

Part 1

Definition of Gentzen System **LI**

The proof system **LI** for Intuitionistic Logic as presented here was published by **G. Gentzen** in 1935

It was presented as a particular case of his proof system **LK** for the classical logic

We present now the **original Gentzen** proof system **LI** and then we show how it can be extended to the **original Gentzen** system **LK**

Language of **LI**

Language of **LI** is

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

We add a new symbol $\longrightarrow$ to the language and call it a Gentzen arrow

We **denote**, as before, the finite sequences of formulas by Greek capital letters

$$\Gamma, \Delta, \Sigma, \ldots$$

with indices if necessary

Language of **LI**

Definition Any expression

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta \in \mathcal{F}^*$ and

Δ consists of **at most one formula**

is called a **LI** sequent

We denote the set of all **LI** sequents by *ISQ*, i.e.

$$ISQ = \{\Gamma \longrightarrow \Delta : \ \Delta \text{ consists of } \textbf{at most one formula}\}$$

Axioms of **LI**

Logical Axioms of **LI** consist of any sequent from the set *ISQ* which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma,\ A,\ \Delta\ \longrightarrow\ A$$

for $\Gamma, \Delta \in \mathcal{F}^*$

Rules of Inference of **LI**

The set inference rules of **LI** is divided into two groups : the **structural rules** and the **logical rules**

There are three **Structural Rules** of **LI**: Weakening, Contraction and Exchange

Weakening structural rule

$$(weak \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak) \quad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}$$

A is called the weakening formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Contraction structural rule

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

A is called the contraction formula

Remember that Δ contains at most one formula

The case below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

Rules of Inference of **LI**

Exchange structural rule

$$(exch \to)\quad \frac{\Gamma_1,\ A, B,\ \Gamma_2\ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2\ \longrightarrow\ \Delta}$$

Remember that Δ contains at most one formula

The rule below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\to exch)\quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}.$$

Rules of Inference of **LI**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Disjunction rules

$$(\longrightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}$$

$$(\longrightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)}$$

$$(\cup \longrightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Gentzen System **LI**

Negation rules

$$(\neg \to) \quad \frac{\Gamma \longrightarrow A}{\neg A,\ \Gamma \longrightarrow}$$

$$(\to \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}$$

We define the Gentzen System **LI** as

$$\mathbf{LI} = (\mathcal{L},\ ISQ,\ LA,\ \text{Structural rules},\ \text{Logical rules})$$

LK - Original Gentzen system
for Classical Propositional Logic

Classical Gentzen System **LK**

Language of **LK**

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \quad \text{and} \quad \mathcal{E} = SQ$$

for

$$SQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^*\}$$

Axioms of LK any sequent of the form

$$\Gamma_1, \ A, \Gamma_2 \quad \longrightarrow \quad \Gamma_3, \ A, \ \Gamma_4$$

Classical Gentzen System **LK**

Rules of inference of LK

1. We adopt all rules of **LI** with **no restriction** that the sequence Δ in the succedent of the sequence is at most one formula

2. We add the following structural rules to the system **LI**

Contraction rule

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

2. We add one more

$$(\rightarrow exch) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}$$

Classical Gentzen System **LK**

Observe that the added rules become obsolete in **LI**

The rules of inference of **LK** are hence as follows

Weakening Structural Rule

$$(weak \rightarrow)\quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak)\quad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, A}$$

Contraction Structural Rule

$$(contr \rightarrow)\quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow contr)\quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

Classical Gentzen System **LK**

Exchange Structural Rule

$$(exch \rightarrow) \quad \frac{\Gamma_1,\ A, B,\ \Gamma_2 \ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2 \ \longrightarrow\ \Delta}$$

$$(\rightarrow exch) \quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}$$

Classical Gentzen System **LK**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta}$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow \Delta, A \quad ; \quad \Gamma \longrightarrow \Delta, B, \Delta}{\Gamma \longrightarrow \Delta, (A \cap B)}$$

Disjunction rules

$$(\rightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta, A, B}{\Gamma \longrightarrow \Delta, (A \cup B)}$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Classical Gentzen System **LK**

Implication rules

$$(\longrightarrow\Rightarrow) \quad \frac{A,\ \Gamma\ \longrightarrow\ \Delta,\ B}{\Gamma\ \longrightarrow\ \Delta,\ (A \Rightarrow B)}$$

$$(\Rightarrow\longrightarrow) \quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A \quad ; \quad B,\ \Gamma\ \longrightarrow\ \Delta}{(A \Rightarrow B),\ \Gamma\ \longrightarrow\ \Delta}$$

Classical Gentzen System **LK**

Negation rules

$$(\neg \longrightarrow) \quad \frac{\Gamma \longrightarrow \Delta, A}{\neg A, \Gamma \longrightarrow \Delta}$$

$$(\longrightarrow \neg) \quad \frac{A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg A}$$

We define formally

$$\mathbf{LK} = (\mathcal{L}, SQ, \ \ LA, \quad \text{Structural rules}, \ \ \text{Logical rules})$$

Gentzen Sequent Calculus **LI** for Intuitionistic Logic
Part 2

Decomposition Trees in **LI**

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL**

In all systems the proof search procedure consists of building the **decomposition trees**

Remark 1

In **RS** the decomposition tree $\mathbf{T}_A$ of any formula A is always unique

Remark 2

In **GL** the "blind search" defines, for any formula A a **finite number** of decomposition trees,

Nevertheless, it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules

Decomposition Trees in **LI**

Remark 3

In **LI** the structural rules play a **vital role** in the proof construction and hence, in the proof search

The fact that a given **decomposition tree** ends with an non-axiom leaf **does not always imply** that does not exist

It might only imply that our search strategy was **not good**

The problem of **deciding** whether a given formula A **does, or does not** have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic

Examples

Example 1

Determine] whether

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Observe that

If we find a decomposition tree of A in **LI** such that all its leaves are axiom, we have a proof , i.e

$$\vdash_{\mathbf{LI}} A$$

If all possible decomposition trees have a non-axiom leaf then the proof of A i n **LI** does not exist, i.e.

$$\nvdash_{\mathbf{LI}} A$$

Examples

Consider the following decomposition tree **$T1_A$**

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(\neg A \cap \neg B),\ (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A,\ \neg B,\ (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B,\ (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B,\ (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge (\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad B \longrightarrow B$$

$$non - axiom \qquad\qquad axiom$$

Examples

The tree **$T1_A$** has a non-axiom leaf, so it does not constitute a proof in **LI**

Observe that the decomposition tree in **LI** is not always unique

Hence this fact does not yet prove that a **proof** of A does not exist

Consider the following decomposition tree **$T2_A$**

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$A, \neg A, \neg B \longrightarrow$	$B, \neg A, \neg B \longrightarrow$
$\mid (exch \longrightarrow)$	$\mid (exch \longrightarrow)$
$\neg A, A, \neg B \longrightarrow$	$B, \neg B, \neg A \longrightarrow$
$\mid (\neg \longrightarrow)$	$\mid (exch \longrightarrow)$
$A, \neg B \longrightarrow A$	$\neg B, B, \neg A \longrightarrow$
axiom	$\mid (\neg \longrightarrow)$
	$B, \neg A \longrightarrow B$; *axiom*

Examples

All leaves of $\mathbf{T2_A}$ are axioms and hence

$\mathbf{T2_A}$ is a **a proof** in **LI**

Hence we proved that

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Examples

Example 2: Show that

1. $\vdash_{\mathbf{LI}} (A \Rightarrow \neg\neg A)$

2. $\nvdash_{\mathbf{LI}} (\neg\neg A \Rightarrow A)$

Solution of **1.**

We construct some, or all decomposition trees of

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

The tree $\mathbf{T}_A$ that ends with all axioms leaves is a proof of A in **LI**

Examples

We construct $\mathbf{T}_A$ as follows

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$A \longrightarrow \neg\neg A$$

$$\mid (\longrightarrow \neg)$$

$$\neg A, A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow A$$

$$\textit{axiom}$$

All leaves of $\mathbf{T}_A$ are axioms what proves that we have found a proof

We **don't need** to construct any other decomposition trees.

Examples

Solution of **2.**

In order to prove that

$$\nvdash_{\mathbf{LI}} \quad (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

and show that each of them has an non-axiom leaf

Examples

Here is **$\mathbf{T1}_A$**

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

one of 2 choices

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

one of 3 choices

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

one of 3 choices

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

one of 2 choices

$$A \longrightarrow$$

non – axiom

Here is $\mathbf{T2}_A$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$\mid (\longrightarrow\Rightarrow)$ *one of 2 choices*

$$\neg\neg A \longrightarrow A$$

$\mid (contr \longrightarrow)$ *second of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$\mid (\longrightarrow weak)$ *first of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow$$

$\mid (\neg \longrightarrow)$ *first of 2 choices*

$$\neg\neg A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *one of 2 choices*

$$A, \neg\neg A \longrightarrow$$

$\mid (exch \longrightarrow)$ *one of 2 choices*

$$\neg\neg A, A \longrightarrow$$

$\mid (\neg \longrightarrow)$*one of 2 choices*

$$A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *first of 2 choices*

$$A, A \longrightarrow$$

non – axiom

Structural Rules

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more **complicated trees**

This is due to the presence of structural rules "blind" application of the rule $(contr \to)$ gives always an infinite number of decomposition trees

In order to decide that none of them will produce a proof we need some **extra knowledge** about patterns of their construction, or just simply about the number o useful of application of **structural rules** within the proofs.

Structural Rules

In this case we can just make an "**external**" **observation** that the our first tree $\mathbf{T1}_A$ is in a sense a minimal one

It means that all other trees would only **complicate** this one in an inessential way, i.e. the we will never produce a tree with all axioms leaves

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its **correctness** is needed and that requires some extra knowledge

Within the scope of this book we accept the "external explanation as a **sufficient solution**, provided its correctness had been proved elsewere

Structural Rules

As we can see from the above examples the structural rules and especially the $(contr \longrightarrow)$ rule **complicates** the proof searching task.

Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules

They also are as we have proved, **complete** with respect to classical semantics.

The original Gentzen system **LK** which does contain the structural rules is also, as proved by Gentzen, **complete**

Structural Rules

Hence all three classical proof system **RS, GL, LK** are equivalent

This proves that the structural rules can be **eliminated** from the system **LK**

A natural question of elimination of structural rules from the Intutionistic Gentzen system **LI** arises

The following **example** illustrates the negative answer

Connection Between Classical and Intuitionistic Logics

Here is the **connection** between Intuitionistic logic and the Classical one

Theorem 1

For any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where

$\models A$ means that A is a **classical tautology**

$\vdash_{IS} A$ means that A is Intutionistically provable in any **Intuitionistically complete** proof system IS

Connection Between Classical and Intuitionistic Logics

A Gentzen system **LI** has been proved to be **Intuitionistically complete** so have that the following

Theorem 2

For any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_{\mathbf{LI}} \neg\neg A$$

Example

Example 3

Obviously

$$\models (\neg\neg A \Rightarrow A)$$

so by **Theorem 2** we must have that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg\neg A \Rightarrow A)$$

We are going to prove now that the structural rule $(contr \longrightarrow)$ is **essential** to the existence of the proof, i.e

We show now that the formula We $\neg\neg(\neg\neg A \Rightarrow A)$ is **not provable** in **LI** without the rule $(contr \longrightarrow)$

The following decomposition tree $\mathbf{T}_A$ is a proof of $A = \neg\neg(\neg\neg A \Rightarrow A)$ in **LI** with use of the contraction rule $(contr \longrightarrow)$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg\neg A, A \longrightarrow A$$

axiom

Contraction Rule

Assume now that the Contraction rule $(contr \longrightarrow)$ is **not available**

All possible decomposition trees are as follows

Tree **T1**$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A \longrightarrow$$

$$non-axiom$$

Contraction Rule

The next is $\mathbf{T2}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non - axiom$$

Contraction Rule

The next is $\mathbf{T3}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non-axiom$$

Contraction Rule

The last one is **T4**$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$]$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

non – axiom

Contraction Rule

We have considered all possible decomposition trees that do not involve the Contraction Rule and **none** of them was a proof

This shows that the formula

$$\neg\neg(\neg\neg A \Rightarrow A)$$

is not provable in **LI** without $(contr \longrightarrow)$ rule, i.e. that

Fact

The Contraction Rule **can't be eliminated** from **LI**

Exercise

Use Gentzen system **LI** to **prove** the following

Theorem (Gödel, Gentzen)

A disjunction $(A \cup B)$ is **intuitionistically provable** if and only if either A or B is **intuitionistically provable** i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \quad \text{or} \quad \vdash_I B$$

Proof Search Heuristic Method

Before we define a heuristic method of searching for proof in **LI** let's put together some **observations**

Observation 1: the logical rules of **LI** are similar to those in Gentzen type classical formalizations we examined in previous chapters in a sense that each of them introduces a logical connective

Observation 2: The process of searching for a proof is, as before a decomposition process in which we use the inverse of logical and structural rules as decomposition rules

Observation 3: We write our proofs in as trees, instead of sequences of expressions, so the proof search process is a process of building a decomposition tree

To facilitate the process we write, as before, the decomposition rules, structural rules in a "tree " form

Proof Search Heuristic Method

We define, as before the notion of decomposable and indecomposable formulas and sequents as follows

Decomposable formula is any formula of the degree ≥ 1

Decomposable sequent is any sequent that contains a decomposable formula

Indecomposable formula is any formula of the degree 0, i.e. any propositional variable

Indecomposable sequent is a sequent formed from indecomposable formulas only.

Proof Search Heuristic Method

Decomposition tree $\mathbf{T_A}$ construction for a given a formula $A \in$ *calF* is as follows

Root of the tree is the sequent $\longrightarrow A$

Given a node n of the tree we **identify** a decomposition rule applicable at this node and write its premisses as the **leaves** of the node n

We stop the decomposition process when we obtain axioms on all branches or all leaves of the tree are i ndecomposable

Proof Search Heuristic Method

Observation 4

We can see from previous examples of **decomposition trees** that the above "blind" construction of all possible trees only leads to **more complicated trees**, due to the presence of structural rules

Observation 5

The "blind" application of structural rule $(contr \longrightarrow)$ gives an **infinite** number of infinite **decomposition trees**

In order to **decide** that none of them would produce a **proof** we need some extra knowledge about **patterns of their construction**, or just simply about the number useful of application of **structural rules** within the search for the proofs

Proof Search Heuristic Method

One can formulate a **deterministic procedure** (and we will do so) giving a finite number of trees

But the proof of correctness of such procedure requires some extra knowledge and theorems to be proved

We are going to discuss here a **motivation** and argue validity of such a **heuristic**

The main point is, as we can see from our examples, that the structural rules and especially the $(contr \rightarrow)$ rule complicate in often useless way the proof searching task

Proof Search Heuristic Method

Observation 6

Our goal while constructing the decomposition tree is to obtain axiom or indecomposable leaves

With respect to this goal the use logical decomposition rules **has a priority** over the use of the structural rules

We use this information while describing the proof search **heuristic**

Proof Search Heuristic Method

Observation 7

All logical decomposition rules $(\circ \rightarrow)$, where $\circ$ denotes any connective, must have a formula we want to decompose as the first formula at the decomposition node

It means that if we want to **decompose** a formula $\circ A$ the node must have a form $\circ A, \Gamma \longrightarrow \Delta$

Remember: order of decomposition is important

Also sometimes it is necessary to decompose a **formula within the sequence** Γ **first**, before decomposing $\circ A$ in order to **find** a proof

Proof Search Heuristic Method

For example, consider two nodes

$$n_1 = \neg\neg A,\ (A \cap B) \longrightarrow B$$

and

$$n_2 = (A \cap B),\ \neg\neg A \longrightarrow B$$

We are going to see that the results of decomposing n_1 and n_2 **differ dramatically**

Let's decompose the node n_1

Observe that the only way to be able to decompose the formula $\neg\neg A$ is to use the rule $(\rightarrow \textit{weak})$ as a **first step**

The **two possible** decomposition trees that starts at the node n_1 are as follows

Proof Search Heuristic Method

First Tree

$$\mathbf{T1}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\cap \rightarrow)$$

$$A, B \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, A, B \longrightarrow$$

non – axiom

Proof Search Heuristic Method

Second Tree

$\mathbf{T2}_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, (A \cap B) \longrightarrow$$

$$\mid (\cap \rightarrow)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

Proof Search Heuristic Method

Let's now decompose the node n_2

Observe that following our **Observation 6** we start by decomposing the formula $(A \cap B)$ by the use of the rule $(\cap \rightarrow)$ as the **first step**

A decomposition tree that starts at the node n_2 is as follows

$$\mathbf{T}_{n_2}$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$\textit{axiom}$$

This proves that the node n_2 is **provable** in **LI**, i.e.

$$\vdash_{\mathbf{LI}} \quad (A \cap B), \neg\neg A \longrightarrow B$$

Proof Search Heuristic Method

Observation 8

The use of structural rules is **important** and **necessary** while we search for proofs

Nevertheless we have to **use them** on the "must" basis and set up some **guidelines** and **priorities** for their use

For example, the use of weakening rule **discharges** the weakening formula, and hence we might **loose an information** that may be essential to finding the **proof**

We should use the weakening rule only when it is **absolutely necessary** for the next decomposition steps

Proof Search Heuristic Method

Hence, the use of weakening rule $(\rightarrow weak)$ **can**, and **should be restricted** to the cases when it leads to **possibility** of the future use of the **negation rule** $(\neg \rightarrow)$

This was the case of the decomposition tree $\mathbf{T1}_{n_1}$

We used the rule $(\rightarrow weak)$ as an necessary step, but it **discharged** too much information and we didn't get a proof, when **proof on this node existed**

Proof Search Heuristic Method

Here is such a proof

T3$_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (exch \longrightarrow)$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

axiom

Proof Search Heuristic Method

Method

For any $A \in \mathcal{F}$ we construct the set of decomposition trees $\mathbf{T}_{\rightarrow A}$ following the rules below.

1. Use first logical rules where applicable.

2. Use $(exch \rightarrow)$ rule to decompose, via logical rules, as many formulas on the left side of $\longrightarrow$ as possible

Remember that the order of decomposition matters! so you have to cover different choices

3. Use $(\rightarrow weak)$ only on a "must" basis and in connection with $(\neg \rightarrow)$ rule

4. Use $(contr \rightarrow)$ rule as the **last recourse** and only to formulas that contain $\neg$ as a main connective

5. Let's call a formula A to which we apply $(contr \rightarrow)$ rule a **a contraction formula** we need to consider are the formulas containing $\neg$ between theirs logical connectives

Proof Search Heuristic Method

7. Within the process of construction of all possible trees use $(contr \rightarrow)$ rule **only** to **contraction formulas**

8. Let C be a **contraction formula** appearing on a node n of the decomposition tree of $\mathbf{T}_{\rightarrow A}$

For any **contraction formula** C, any node n, we apply $(contr \rightarrow)$ rule the the formula C **at most** as many times as the number of sub-formulas of C

If we **find** a tree with all axiom leaves we have a **proof**, i.e.

$$\vdash_{LI} A$$

If **all trees** (finite number) have a non-axiom leaf we have proved that proof of A **does not exist**, i.e.

$$\nvdash_{LI} A$$