cse541
LOGIC FOR COMPUTER SCIENCE

Professor Anita Wasilewska

Spring 2015

LECTURE 11

Chapter 11
Introduction to Intuitionistic Logic

Short History

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as intuitionism

Intuitionism was originated by L. E. J. Brouwer in 1908

The first Hilbert style formalization of the **Intuitionistic logic** formulated as a proof system only, is due to A. Heyting in 1930

We present here a Hilbert style proof system **I** for **Intuitionistic Propositional Logic**

The proof system **I** is equivalent to the Heyting's original formalization

We also discuss a **relationship** between the Intuitionistic and Classical logics

Short History

There have been, of course, several successful attempts at creating **semantics** for the intuitionistic logic, and hence to define formally a notion of the **intuitionistic tautology**

The most known are Kripke models and algebraic models

Kripke models were defined by Kripke in 1964

Algebraic models were initiated by Stone and Tarski in 1937, 1938, respectively

An uniform theory and presentation of **topological** and **algebraic models** was given by Rasiowa and Sikorski in 1964

Hilbert Proof System for Intuitionistic Propositional Logic

Language

We adopt a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}}$$

with the set of formulas denoted by $\mathcal{F}$

Logical Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

Hilbert Proof System for Intuitionistic Propositional Logic

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

where A, B, C are any formulas in $\mathcal{L}$

Rules of inference

We adopt a Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only rule of inference

Proof System I

A proof system

$$\mathbf{I} = (\ \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}},\ \mathcal{F},\ \ \{A1, ..., A11\},\ (MP)\ )$$

is called a Hilbert Style Formalization for Intuitionistic Propositional Logic

The set of axioms $\{A1, ..., A11\}$ is due to Rasiowa (1959)

It differs from Heyting's original set of axioms but they are equivalent

We introduce, as usual, the notion of a formal proof in **I** and denote by

$$\vdash_I A$$

the fact that a formula A has a formal proof in **I** and we say that the formula A is **intuitionistically provable**

Completeness Theorem

There are several ways one can define a **semantics** for the intuitionistic logic

Define a semantics for the intuitionistic logic means to define the semantics for the original Heyting proof system and **prove the Completeness Theorem** for it under this semantics

The same applies to any other equivalent proof system, in particular for our proof system **I**

Completeness Theorem

The notion of intuitionistic semantics and hence the formal definition of **intuitionistic tautology** will be defined and discussed later

For a moment we denote by

$$\models_I A$$

the fact that A is an **intuitionistic tautology** under some **intuitionistic semantics**

Let's denote by **IS** any proof system **equivalent** to the original Heyting system for Intuitionistic logic

Completeness Theorem for the proof system **IS**
For any formula $A \in \mathcal{F}$,

$$\vdash_{IS} A \quad \text{if and only if} \quad \models_I A$$

Examples of Intuitionistic Tautologies

Of course, all of **Logical Axioms** A1 - A11 of our proof system **I** are Intuitionistic tautologies

Here are some other **classical tautologies** that are also **Intuitionistic tautologies**

1. $(A \Rightarrow A)$

2. $(A \Rightarrow (B \Rightarrow A))$

3. $(A \Rightarrow (B \Rightarrow (A \cap B)))$

4. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

5. $(A \Rightarrow \neg\neg A)$

6. $\neg(A \cap \neg A)$

7. $((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

Examples of Intuitionistic Tautologies

8. $(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$

9. $((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$

10. $((\neg A \cup \neg B) \Rightarrow \neg(A \cap B))$

11. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

12. $((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$

13. $(\neg\neg\neg A \Rightarrow \neg A)$

14. $(\neg A \Rightarrow \neg\neg\neg A)$

15. $(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$

16. $((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$

Examples of NOT Intuitionistic Tautologies

The following classical tautologies **are not** intuitionistic tautologies

17. $(A \cup \neg A)$

18. $(\neg\neg A \Rightarrow A)$

19. $((A \Rightarrow B) \Rightarrow (\neg A \cup B))$

20. $(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$

21. $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$

22. $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$

23. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$,

Homework Exercises

The general idea of **algebraic models** for the intuitionistic logic is defined in terms of **Pseudo-Boolean Algebras** in the following way

A formula A is said to be an **intuitionistic tautology** if and only if $v \models A$, for all v and **all Pseudo-Boolean Algebras**, where v maps the propositional variable VAR into the universe of a **Pseudo-Boolean Algebra**

Definition

A formula A is an **intuitionistic tautology** if and only if it is true in all **Pseudo-Boolean Algebras** under all possible variable assignments v

Homework Exercises

The 3 element Heyting algebra **H** as defined in the section "Some three valued logics" is an example of a 3 element **Pseudo-Boolean Algebra**

Exercise 1

Show that the 3 element Heyting algebra **H** is a model for all logical axioms **A1- A11** and all of the formulas **1-16**, i.e. show that they are all **H- tautologies**

Exercise 2

Find for which of the formulas **17 - 23** the 3 element Heyting algebra acts as a **counter-model**

Connection Between Classical and Intuitionistic Logics

The first connection is quite obvious.

It was proved by Rasiowa and Sikorski in 1964 that by adding the axiom

A12 $(A \cup \neg A)$

to the set of axioms of our system **I** we obtain a Hilbert proof system **C** that is **complete** with respect to classical semantics

This proves the following.

Theorem 1

Every formula that is intuitionistically derivable is also classically derivable, i.e. the implication

$$\text{If} \quad \vdash_I A \quad \text{then} \quad \vdash_C A$$

holds for any $A \in \mathcal{F}$

Connection Between Classical and Intuitionistic Logics

We write

$$\models A$$

and

$$\models_I A$$

to denote that A is a **classical** and **intuitionistic** tautology, respectively.

As both proof systems **I** and **C** are complete under respective semantics, we can re-write Theorem 1 as the following **relationship** between classical and intuitionistic **tautologies**

Theorem 2 For any formula $A \in \mathcal{F}$,

$$\text{If } \models_I A, \text{ then } \models A$$

Connection Between Classical and Intuitionistic Logics

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa

The following has been proved by Glivenko in 1929 in terms of provability as the semantics for Intuitionisctic Logic didn't yet exist

Theorem 3 (Glivenko)

For any formula $A \in \mathcal{F}$,

A is **classically** provable if and only if $\neg\neg A$ is an **intuitionistically** provable, i.e.

$$\vdash_C A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol $\vdash_C$ for classical provability in a complete classical proof system

Connection Between Classical and Intuitionistic Logics

The following has been proved by Tarski in 1938 together with a **definition of algebraic semantics** for Intuitionistic Logic

Theorem 4 (Tarski)

For any formula $A \in \mathcal{F}$,

A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \quad \text{if and only if} \quad \models_I \neg\neg A$$

Connection Between Classical and Intuitionistic Logics

The following relationships were proved by Gödel in 1331.

Theorem 5 (Gödel)

For any formulas $A, B \in \mathcal{F}$,

a formula $(A \Rightarrow \neg B)$ is **classically provable** if and only if it is **intuitionistically provable**, i.e.

$$\vdash_C (A \Rightarrow \neg B) \quad \text{if and only if} \quad \vdash_I (A \Rightarrow \neg B)$$

Theorem 6 (Gödel)

For any formula $A, B \in \mathcal{F}$,

If A contains no connectives except $\cap$ and $\neg$, then A i is **classically provable** if and only if it is **intuitionistically provable**

Connection Between Classical and Intuitionistic Logics

By the **Completeness Theorems** for classical and intuitionisctic logics we get the following equivalent **semantic** form of Gödel' s **Theorems 5, 6**

Theorem 6

A formula $(A \Rightarrow \neg B)$ is a **classical tautology** if and only if it is an **intuitionistic tautology**, i.e.

$$\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)$$

Theorem 7

If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology

On intuitionistically derivable disjunction

In a **classical logic** it is possible for the disjunction $(A \cup B)$ to be a **tautology** when neither A nor B is a **autology**

The tautology $(A \cup \neg A)$ is the simplest example

This does not hold for the **intuitionistic logic**

This fact was stated without the proof by Gödel in 1931 and **proved** by Gentzen in 1935 via his proof system **LI** which is presented and discussed in chapter 12 and Lecture 15

On intuitionistically derivable disjunction

Remember that Gödel and Gentzen meant by **intuitionistic logic** a Heyting proof system or any other proof system (like the one defined by Gentzen) equivalent with it

The following theorem was announced without the proof by Gödel in 1931 and proved by Gentzen in 1934

Theorem 8 (Gödel, Gentzen)

A disjunction $(A \cup B)$ is **intuitionistically provable** if and only if either A or B is **intuitionistically provable** i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \quad \text{or} \quad \vdash_I B$$

We obtain, via the **Completeness Theorem** the following equivalent semantic version of the above

Theorem 9

A disjunction $(A \cup B)$ is **intuitionistic tautology** if and only if either A or B is **intuitionistic tautology**, i.e.

$$\models_I (A \cup B) \quad \text{if and only if} \quad \models_I A \quad \text{or} \quad \models_I B$$