cse541
LOGIC FOR COMPUTER SCIENCE

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LECTURE 10a

Chapter 10
CLASSICAL AUTOMATED PROOF SYSTEMS

PART 3: GENTZEN SYSTEMS

Gentzen Sequent Calculus GL

The proof system **GL** for the classical propositional logic presented now is a version of the original **Gentzen** (1934) systems **LK**.

A **constructive** proof of the Completeness Theorem for the system **GL** is very similar to the proof of the Completeness Theorem for the system **RS**

Expressions of the system arec like in the original Gentzen system **LK** are Gentzen **sequents**

Hence we use also a name Gentzen sequent calculus for it

Gentzen Sequent Calculus GL

Language of **GL**

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$$

We add a new symbol to the alphabet $\longrightarrow$ called a Gentzen arrow

The **sequents** are built out of finite sequences (empty included) of formulas, i.e. elements of $\mathcal{F}^*$, and the additional symbol $\longrightarrow$

We **denote**, as in the **RS** system, the finite sequences of formulas by Greek capital letters

$$\Gamma, \Delta, \Sigma, \ldots$$

with indices if necessary

Gentzen Sequents

Definition Any expression

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta \in \mathcal{F}^*$ is called a sequent

Intuitively, we interpret semantically a sequent

$$A_1, ..., A_n \longrightarrow B_1, ..., B_m$$

where $n, m \geq 1$, as a formula

$$(A_1 \cap ... \cap A_n) \Rightarrow (B_1 \cup ... \cup B_m)$$

Gentzen Sequents

The sequent

$$A_1, ..., A_n \longrightarrow$$

(where $m \geq 1$) means that $A_1 \cap ... \cap A_n$ yields a **contradiction**

The sequent

$$\longrightarrow B_1, ..., B_m$$

(where $m \geq 1$) means semantically $T \Rightarrow (B_1 \cup ... \cup B_m)$

The empty sequent

$$\longrightarrow$$

means a **contradiction**

Gentzen Sequents

Given non empty sequences Γ, Δ

We denote by σ_Γ any conjunction of all formulas of Γ

We denote by δ_Δ any disjunction of all formulas of Δ

The **intuitive semantics** of a non- empty sequent $\Gamma \longrightarrow \Delta$ is

$$\Gamma \longrightarrow \Delta \equiv (\sigma_\Gamma \Rightarrow \delta_\Delta)$$

Formal Semantics

Formal semantics for sequents of **GL** is defined as follows

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment and v^* its extension to the set of formulas $\mathcal{F}$

We **extend** v^* to the set

$$SQ = \{\, \Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^* \,\}$$

of all sequents as follows

For any sequent $\Gamma \longrightarrow \Delta \in SQ$

$$v^*(\Gamma \longrightarrow \Delta) \;=\; v^*(\sigma_\Gamma) \Rightarrow v^*(\delta_\Delta)$$

Formal Semantics

In the case when $\Gamma = \emptyset$ or $\Delta = \emptyset$ we **define**

$$v^*(\longrightarrow \Delta) = (T \Rightarrow v^*(\delta_\Delta))$$

$$v^*(\Gamma \longrightarrow) = (v^*(\sigma_\Gamma) \Rightarrow F)$$

The sequent $\Gamma \longrightarrow \Delta$ is **satisfiable** if there is a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that

$$v^*(\Gamma \longrightarrow \Delta) = T$$

Formal Semantics

Model for $\Gamma \longrightarrow \Delta$ is any v such that

$$v^*(\Gamma \longrightarrow \Delta) = T$$

We write it $v \models \Gamma \longrightarrow \Delta$

Counter- model is any v such that

$$v^*(\Gamma \longrightarrow \Delta) = F$$

We write it $v \not\models \Gamma \longrightarrow \Delta$

Tautology is any sequent $\Gamma \longrightarrow \Delta$ such that

$v^*(\Gamma \longrightarrow \Delta) = T$ for all truth assignments $v : VAR \longrightarrow \{T, F\}$

We write it

$$\models \Gamma \longrightarrow \Delta$$

Example

Example

Let $\Gamma \longrightarrow \Delta$ be a sequent

$$a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

The truth assignment v for which

$$v(a) = T \quad \text{and} \quad v(b) = T$$

is a **model** for $\Gamma \longrightarrow \Delta$ as shows the following computation

$$v^*(a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)) = v^*(\sigma_{\{a,(b\cap a)\}}) \Rightarrow v^*(\delta_{\{\neg b,(b\Rightarrow a)\}})$$

$$= v(a) \cap (v(b) \cap v(a)) \Rightarrow \neg v(b) \cup (v(b) \Rightarrow v(a))$$

$$= T \cap T \cap T \Rightarrow \neg T \cup (T \Rightarrow T) = T \Rightarrow (F \cup T) = T \Rightarrow T = T$$

Example

Observe that the truth assignment v for which

$$v(a) = T \quad \text{and} \quad v(b) = T$$

is the **only one** for which

$$v^*(\Gamma) = v^*(a, (b \cap a) = T$$

and we proved that it is a **model** for

$$a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

It is hence **impossible** to find v which would **falsify it**, what proves that

$$\models a, (b \cap a) \longrightarrow \neg b, (b \Rightarrow a)$$

Gentzen System **GL**

Definition of **GL**

Logical Axioms LA

We adopt as an axiom any sequent of variables (positive literals) which contains a propositional variable that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma'_1, a, \Gamma'_2 \longrightarrow \Delta'_1, a, \Delta'_2$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$

Gentzen System **GL**

Inference rules of **GL**

Let $\Gamma', \Delta' \in VAR^*$ and $\Gamma, \Delta \in \mathcal{F}^*$

Conjunction rules

$$(\cap \rightarrow) \quad \frac{\Gamma', \; A, B, \; \Gamma \; \longrightarrow \; \Delta'}{\Gamma', \; (A \cap B), \; \Gamma \; \longrightarrow \; \Delta'}$$

$$(\rightarrow \cap) \quad \frac{\Gamma \; \longrightarrow \; \Delta, \; A, \; \Delta' \quad ; \quad \Gamma \; \longrightarrow \; \Delta, \; B, \; \Delta'}{\Gamma \; \longrightarrow \; \Delta, \; (A \cap B) \; \Delta'}$$

Gentzen System **GL**

Disjunction rules

$$(\longrightarrow \cup) \quad \frac{\Gamma \longrightarrow \Delta,\ A, B,\ \Delta'}{\Gamma \longrightarrow \Delta,\ (A \cup B),\ \Delta'}$$

$$(\cup \longrightarrow) \quad \frac{\Gamma',\ A,\ \Gamma \longrightarrow \Delta' \quad ; \quad \Gamma',\ B,\ \Gamma \longrightarrow \Delta'}{\Gamma',\ (A \cup B),\ \Gamma \longrightarrow \Delta'}$$

Gentzen System **GL**

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{\Gamma',\ A,\ \Gamma\ \longrightarrow\ \Delta,\ B,\ \Delta'}{\Gamma', \Gamma\ \longrightarrow\ \Delta,\ (A \Rightarrow B),\ \Delta'}$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma', \Gamma\ \longrightarrow\ \Delta,\ A,\ \Delta' \quad ; \quad \Gamma',\ B,\ \Gamma\ \longrightarrow\ \Delta, \Delta'}{\Gamma',\ (A \Rightarrow B),\ \Gamma\ \longrightarrow\ \Delta, \Delta'}$$

Gentzen System **GL**

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma', \Gamma \longrightarrow \Delta, A, \Delta'}{\Gamma', \neg A, \Gamma \longrightarrow \Delta, \Delta'}$$

$$(\rightarrow \neg) \quad \frac{\Gamma', A, \Gamma \longrightarrow \Delta, \Delta'}{\Gamma', \Gamma \longrightarrow \Delta, \neg A, \Delta'}$$

Gentzen System **GL**

We define the Gentzen System **GL**

$$\mathbf{GL} = (\ \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}},\ \ SQ,\ \ LA,\ \ \mathcal{R}\ )$$

for

$$\mathcal{R} = \{(\cap \longrightarrow),\ (\longrightarrow \cap),\ (\cup \longrightarrow),\ (\longrightarrow \cup),\ (\Rightarrow\longrightarrow),\ (\longrightarrow\Rightarrow)\}$$

$$\cup \{(\neg \longrightarrow),\ (\longrightarrow \neg)\}$$

We write, as usual,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta$$

to denote that $\Gamma \longrightarrow \Delta$ has a formal proof in **GL**

A formula $A \in \mathcal{F}$, has a proof in **GL** if the sequent $\longrightarrow A$ has a proof in **GL**, i.e.

$$\vdash_{\mathbf{GL}} A \quad \text{if ad only if} \quad \longrightarrow A$$

Gentzen System **GL**

We consider, as we did with **RS** the proof trees for **GL**, i.e. we define

A **proof tree**, or **GL**-proof of $\Gamma \longrightarrow \Delta$ is a tree

$$\mathbf{T}_{\Gamma \longrightarrow \Delta}$$

of sequents satisfying the following conditions:

1. The topmost sequent, i.e **the root** of $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ is $\Gamma \longrightarrow \Delta$

2. All **leafs** are axioms

3. The **nodes** are sequents such that each sequent on the tree follows from the ones immediately preceding it by one of the rules.

Gentzen System **GL**

Exercise 1

We define, in a similar way as in **RS** the **GL** the notions of decomposable and indecomposable sequences, the decomposition rules and the decomposition tree

Remark

The proof search in **GL** as defined by the decomposition tree for a given formula A **is not always unique**

We show it on an example on the next slide

Example

A tree-proof in **GL** of the de Morgan Law

$$\longrightarrow\ (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(a \cap b)\ \longrightarrow\ (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b)\ \longrightarrow\ \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b)\ \longrightarrow\ \neg a$$

$$\mid (\longrightarrow \neg)$$

$$b, a, \neg(a \cap b)\ \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$b, a \longrightarrow\ (a \cap b)$$

$$\bigwedge (\longrightarrow \cap)$$

$$b, a \longrightarrow\ a \qquad\qquad b, a \longrightarrow\ b$$

Example

Here is another tree-proof in **GL** of the de Morgan Law

$$\longrightarrow\ (\neg(a\cap b)\Rightarrow(\neg a\cup\neg b))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg(a\cap b)\ \longrightarrow\ (\neg a\cup\neg b)$$

$$\mid (\longrightarrow\cup)$$

$$\neg(a\cap b)\ \longrightarrow\ \neg a,\neg b$$

$$\mid (\longrightarrow\neg)$$

$$b,\neg(a\cap b)\ \longrightarrow\ \neg a$$

$$\mid (\neg\longrightarrow)$$

$$b\ \longrightarrow\ \neg a,(a\cap b)$$

$$\bigwedge(\longrightarrow\cap)$$

$$b\longrightarrow\ \neg a, a \qquad\qquad b\longrightarrow\ \neg a, b$$

$$\mid(\longrightarrow\neg) \qquad\qquad \mid(\longrightarrow\neg)$$

$$b,a\longrightarrow\ a \qquad\qquad b,a\longrightarrow\ b$$

Exercises

Exercise 2

Write all other proofs in **GL** of

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

Exercise 3

Find a formula which has a unique decomposition tree

Exercise 4

Describe for which kind of formulas the decomposition tree is unique

Exercise 5

Formulate and **prove** a Decomposition Tree Theorem for **GL** that corresponds to the similar theorem for **RS**

Strong Soundness and Completeness of **GL**

Exercise 6

Prove, in a similar way as we did in the case of **RS** type systems, the **Strong Soundness** of **GL**

Exercise 7

Prove, in a similar way as we did in the case of **RS**, the **Completeness Theorem** for **GL**, i.e.

Completeness Theorem

For any sequent $\Gamma \longrightarrow \Delta \in SQ$

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta \quad \text{if and only if} \quad \models \Gamma \longrightarrow \Delta$$