CSE541 Take-Home FINAL Spring 2013
100pts + 20 extra points

NAME **ID:**

Final is due May 22 , or ANY DAY before. Slid it under my door when you are ready- and send me e-mail. "The sooner the better" - this is my advise!

QUESTION 1 Consider a propositional language $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$ with a set $\mathcal{F}$ of formulas.

Let $\mathbf{T} \subseteq \mathcal{F}$ be the set of all propositional TAUTOLOGIES under the classical semantics.

Let S be a **COMPLETE Hilbert proof system** with for a classical propositional logic with the language $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$, i.e.

$$\mathbf{T} = \{A \in \mathcal{F} : \vdash_S A\}$$

Prove that for any $A, B \in F$,

$$Cn(\{A\}) \cap Cn(\{B\}) = Cn(\{(A \cup B)\}),$$

where for any Let $X \subseteq F$ we define

$$Cn(X) = \{A \in F : X \vdash_S A\}$$

QUESTION 2

Let S be the proof system from Question 1.

We define **two binary relations** on $\mathcal{F}$ as follows. For any $A, B \in \mathcal{F}$,

$A \leq_S B$ if and only if $\vdash_S (A \Rightarrow B)$ and

$A \leq_{\mathbf{T}} B$ if and only if $\models (A \Rightarrow B)$

1. PROVE that $\leq_S = \leq_{\mathbf{T}}$

2. Prove that $\leq = \leq_S = \leq_{\mathbf{T}}$ is a quasi order relation (reflexive and transitive).

3. Define a non-classical logic semantics M for $\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}}$ such that the binary relation $\leq_M$ on $\mathcal{F}$ defined as

$$A \leq_M B \quad \text{if and only if} \quad \models_M (A \Rightarrow B)$$

is NOT a quasi order.
$\models_M A$ reads: a formula A is a tautology under semantics M

QUESTION 3

Extend the **Proof 2** of Completeness Theorem to the Language that includes $\cap$ and $\cup$.

Write specifically all formulas provability of which must be assumed in the system S with extended language.

Hint You must extend the proof of **Property of** v **Lemma** to the language with $\cap$ and $\cup$.

EXTRA CREDIT Let $\leq = \leq_S = \leq_{\mathbf{T}}$, we define a binary relation $\approx$ on $\mathcal{F}$ as follows.

$$A \approx B \quad \text{iff} \quad A \leq B \text{ and } B \leq A.$$

1. PROVE that $\approx$ ia an equivalence relation (reflexive, symmetric and transitive).

2. Find $[(a \cup \neg a)]$, $[(a \cap \neg a)]$ and $[a]$, where $[A]$ denotes an equivalence class of $\approx$ with a representant A.