CSE541   Take-Home FINAL   Spring 2013
100pts + 20 extra points

NAME                                    ID:

Final is due May 22, or ANY DAY before. Slid it under my door when you are ready- and send me e-mail. "The sooner the better" - this is my advise!

QUESTION 1  Consider a propositional language $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ with a set $\mathcal{F}$ of formulas.

Let $\mathbf{T} \subseteq \mathcal{F}$ be the set of all propositional TAUTOLOGIES under the classical semantics.

Let $\mathbf{S}$ be a COMPLETE Hilbert proof system with for a classical propositional logic with the language $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$, i.e.

$$\mathbf{T} = \{ A \in \mathcal{F} : \vdash_{\mathbf{S}} A \}$$

Prove that for any $A, B \in \mathcal{F}$,

$$\text{Cn}(\{A\}) \cap \text{Cn}(\{B\}) = \text{Cn}(\{(A \cup B)\})$$

where for any Let $X \subseteq \mathcal{F}$ we define

$$\text{Cn}(X) = \{ A \in \mathcal{F} : X \vdash_{\mathbf{S}} A \}$$

QUESTION 2

Let $\mathbf{S}$ be the proof system from Question 1.

We define two binary relations on $\mathcal{F}$ as follows. For any $A, B \in \mathcal{F}$,

$$A \leq_{\mathbf{S}} B \quad \text{if and only if} \quad \vdash_{\mathbf{S}} (A \Rightarrow B) \quad \text{and}$$

$$A \leq_{\mathbf{T}} B \quad \text{if and only if} \quad \models (A \Rightarrow B)$$

1. PROVE that $\leq_{\mathbf{S}} = \leq_{\mathbf{T}}$

2. Prove that $\leq_{\mathbf{S}} = \leq_{\mathbf{T}}$ is a quasi order relation (reflexive and transitive).
3. Define a non-classical logic semantics $M$ for $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$ such that the binary relation $\leq_M$ on $\mathcal{F}$ defined as

$$A \leq_M B \quad \text{if and only if} \quad \models_M (A \Rightarrow B)$$

is NOT a quasi order.

$\models_M A$ reads: a formula $A$ is a tautology under semantics $M$

**QUESTION 3**

Extend the Proof 2 of Completeness Theorem to the Language that includes $\cap$ and $\cup$.

Write specifically all formulas provability of which must be assumed in the system $S$ with extended language.

**Hint** You must extend the proof of Property of $v$ Lemma to the language with $\cap$ and $\cup$.

**EXTRA CREDIT** Let $\leq = \leq_S = \leq_T$, we define a binary relation $\approx$ on $\mathcal{F}$ as follows.

$$A \approx B \quad \text{iff} \quad A \leq B \text{ and } B \leq A.$$ 

1. PROVE that $\approx$ ia an equivalence relation (reflexive, symmetric and transitive).

2. Find $[(a \cup \neg a)]$, $[(a \cap \neg a)]$ and $[a]$, where $[A]$ denotes an equivalence class of $\approx$ with a representant $A$. 

2