HW – 1 (Properties of Consequence Operation $Cn_s(\Gamma)$)

(1) Prove the **Monotonicity Property**.

In order to prove that $Cn_s(\Gamma) \subseteq Cn_s(\Delta)$ (providing $\Gamma \subseteq \Delta$), we should prove that, for any arbitrary expression $B \in \mathcal{E}$ if $B \in Cn_s(\Gamma)$, then $B \in Cn_s(\Delta)$.

Now, according to the definition of $Cn_s(\Gamma)$, we can say that, $B \in Cn_s(\Gamma)$ means that there exists a sequence $A_1, A_2, ..., A_n$ of expressions from $\mathcal{E}$, such that

$$A_1 \in LA \bigcup \Gamma \text{ and } A_n = B$$

And for each $1 < i \le n$, either $A_i \in LA \bigcup \Gamma$ or A_i is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference. Now, we can say that since $\Gamma \subseteq \Delta$, the same sequence $A_1, A_2, ..., A_n$ can be used in order to prove that $B \in Cn_s(\Delta)$. Because, in this sequence $A_1 \in LA \bigcup \Gamma \xrightarrow{\Gamma \subseteq \Delta} A_1 \in LA \bigcup \Delta$ and for each $1 < i \le n$, either $A_i \in LA \bigcup \Gamma \xrightarrow{\Gamma \subseteq \Delta} A_i \in LA \bigcup \Delta$ or A_i is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference (providing that rules are the same for both hypotheses set Γ and Δ). So, since there exists a sequence $A_1, A_2, ..., A_n$ of expression from $\mathcal{E}$ with properties in definition of $Cn_s(\Delta)$, we can say that $B \in Cn_s(\Delta)$.

(2) Prove the **Transitivity Property**.

By considering definition of $Cn_s(\Gamma)$, we have:

1. $\Gamma_1 \subseteq Cn_s(\Gamma_2)$ means that for any $A \in \Gamma_1$, there exists a sequence of expressions, called $(*)$ sequence, $A_1, A_2, ..., A_n = A$ such that $A_1 \in LA \bigcup \Gamma_2$ and for $1 < i \le n$, either $A_i \in LA \bigcup \Gamma_2$ or A_i is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference.
2. $\Gamma_2 \subseteq Cn_s(\Gamma_3)$ means that for any $A \in \Gamma_2$, there exists a sequence of expressions, $A_1^{'}, A_2^{'}, ..., A_n^{'} = A$ such that $A_1^{'} \in LA \bigcup \Gamma_3$ and for $1 < i \le n$, either $A_i^{'} \in LA \bigcup \Gamma_3$ or $A_i^{'}$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference.

So, if in sequence $(*)$, for any of expression, A_i which $A_i \in \Gamma_2$ we replace a sequence $A_1^{'}, A_2^{'}, ..., A_m^{'} = A_i$ such that $A_1^{'} \in LA \bigcup \Gamma_3$ and for $1 < i \le m$, either $A_i^{'} \in LA \bigcup \Gamma_3$ or $A_i^{'}$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference, then a new sequence, such as $A_1^{''}, A_2^{''}, ..., A_k^{''}$ will be created such that $A_k^{''} = A \in \Gamma_1$ and $A_1^{''} \in LA \bigcup \Gamma_3$ and for $1 < i \le k$, $A_i^{''} \in LA \bigcup \Gamma_3$ or $A_i^{''}$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference. So, $\Gamma_1 \subseteq Cns(\Gamma_3)$.

(3) Prove the **Finiteness Property**.

In order to prove this property, we should prove followings:

(1) If there is a finite subset Γ_0 of Γ such that $A \in Cn_s(\Gamma_0)$, then $A \in Cn_s(\Gamma)$.

(2) If $A \in Cn_s(\Gamma)$, then there is a finite subset Γ_0 of Γ such that $A \in Cn_s(\Gamma_0)$.

Proving (1):

Since it is assumed that $\Gamma_0 \subseteq \Gamma$, then according to the monotonicity property, $Cn_s(\Gamma_0) \subseteq Cn_s(\Gamma)$. So, if $A \in Cn_s(\Gamma_0)$, then $A \in Cn_s(\Gamma)$.

Proving (2):

$A \in Cn_s(\Gamma)$ means that there exists a ***finite*** sequence $A_1, A_2, ..., A_n = A$ such that $A_1 \in LA \bigcup \Gamma$ and for any $1 < i \leq n$, $A_i \in LA \bigcup \Gamma$ or A_i is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference. So, some of the expressions of $A_1, A_2, ..., A_n = A$ are from LA and some of them are from Γ and some of them are direct consequences of some of the preceding expressions by virtue of one of the rules of inference. Now, if we call the set of expressions from Γ, used in this sequence as Γ_0, then according to the definition we can say that $\Gamma_0 \vdash_s A$. So, since number of expressions in sequence $A_1, A_2, ..., A_n = A$ is finite, then set Γ_0 is finite. So, there exists a finite subset of Γ (called Γ_0) such that $A \in Cn_s(\Gamma_0)$.