QUESTION 1

(a) Give an example (by name) of three non-classical logics.

Solution: Intuitionistic Logic, Modal Logic S4, S5, and any of CS logics listed below.

(b) (5pts) Give an example (by name) of two logics developed by computer scientists.

Solution: Dynamic logic (Harel 1979) which was created to facilitate the statement and proof of properties of programs.

Temporal Logics which were created for the specification and verification of concurrent programs Harel, Parikh, 1979, 1983 and for a specification of hardware circuits Halpern, Manna and Maszkowski, (1983).

Fuzzy logic, Many valued logics that were created and developed to describe reasoning with incomplete information.

Non-monotonic logics were created by Mc Carthy (1985) and has been shown to be important in other areas. There are applications to logic programming, to planning and reasoning about action, and to automated diagnosis.

QUESTION 2

Let \( A \) be a formula

\[
A = (((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)).
\]

(1) Define a formal language to which the formula \( A \) belongs.

Solution: The language is \( \mathcal{L}_{\{\neg, \cap, \Rightarrow\}} \).

(2) Determine the degree of \( A \) and write down all its sub-formulas of the degree 2.

Solution: The degree of \( A \) is 7.

There is only one sub-formula of the degree 2: \((a \cap \neg c))\).

(3) Determine the following: \( A \in T \), \( A \in C \). You can use the shorthand notation.

Solution of the case \( A \in T \).

Assume \(((a \cap \neg c) \Rightarrow \neg b) \cup a) \Rightarrow (c \cup b)) = F \). This is possible if and only if \(((a \cap \neg c) \Rightarrow \neg b) \cup a) = T \) and \((c \cup b) = F \). This gives as that \( c = F, b = F \). We evaluate \(((a \cap \neg F) \Rightarrow \neg F) \cup a) = T \). This is possible for \( a = T \).

Any truth assignment such that \( a = T, b = F, c = F \) is a counter-model for \( A \), hence \( A \not\in T \).

Solution of the case \( A \in C \).

Any truth assignment such that \( a = T, b = T, c = F \) is a model for \( A \), hence \( A \not\in C \). This is not the only model.

(4) Determine the following: (use a shorthand notation)

(a) \( A \in LT \).

Solution 1: We have proved that \( LT \subseteq T \) and \( A \not\in T \), hence \( A \not\in LT \).
Solution 2: Any truth assignment such that $a = T, b = F, c = F$ is a counter-model for $A$, hence $A \not\in \mathcal{L}_T$.

\textbf{L semantics} is defined as follows.

\begin{tabular}{|c|c|c|}
\hline
\textbf{L Negation} & \textbf{L Disjunction} \\
\hline
$\neg$ & $\cup$ \\
\hline
$F$ & $T$ & $\perp$ & $T$ \\
$T$ & $\perp$ & $F$ & $\perp$ \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{L Conjunction} & \textbf{L-Implication} \\
\hline
$\cap$ & $\Rightarrow$ \\
\hline
$F$ & $F$ & $\perp$ & $T$ & $\perp$ \\
$F$ & $\perp$ & $T$ & $F$ & $\perp$ \\
$T$ & $F$ & $\perp$ & $T$ & $T$ \\
\hline
\end{tabular}

\textbf{QUESTION 3} Write the formula $A$ from Question 2 as a formula of the language $\mathcal{L}_{\{\neg, \cup\}}$, i.e. as a formula $B$ of $\mathcal{L}_{\{\neg, \cup\}}$, such that $A \equiv B$. Write down all logical equivalences you need while solving this problem.

\textbf{Solution:}

\[ (((a \cap \neg c) \Rightarrow \neg b) \cup a) \equiv^{imp} (((a \cap \neg c) \Rightarrow \neg b) \cup (c \cup b)) \equiv^{imp} (((a \cap \neg c) \cup \neg b) \cup (c \cup b)). \]

\textbf{QUESTION 4}

$S$ is the following proof system:

\[ S = ( \mathcal{L}_{\{\Rightarrow, \cup, \neg\}}, F, \ A1, \ (r1), \ (r2) ) \]

\textbf{Axiom}

\textbf{A1} \ (A \Rightarrow (A \cup B)),

\textbf{Rules} of inference:

\begin{align*}
(r1) & \quad \frac{A : B}{A \cup \neg B}, \\
(r2) & \quad \frac{A : (A \cup B)}{B}.
\end{align*}

1. Verify whether $S$ is sound/not sound under classical semantics.

\textbf{Solution} The system is not sound. Take $a, b, c$ such that it evaluates $A = T$ and $B = F$. The premiss $(A \cup B)$ of the rule $(r2)$ is $T$ and the conclusion is $F$.

2. Find a formal proof of $\neg(A \Rightarrow (A \cup B))$ in $S$, i.e. show that $\not\vdash_S \neg(A \Rightarrow (A \cup B))$. 

2
Solution The proof is as follows

$B_1 (A \Rightarrow (A \cup B)),$

$B_2 (A \Rightarrow (A \cup B)),$

$B_3 ((A \Rightarrow (A \cup B)) \cup \neg(A \Rightarrow (A \cup B))),$

$B_4 \neg(A \Rightarrow (A \cup B)).$

3. Does above point 2 prove that $\models \neg(A \Rightarrow (A \cup B))$?

Solution No, the proof used rule (r2) that is not sound.

**QUESTION 5**

$H$ is the following proof system:

\[ H = (\mathcal{L}_{\Rightarrow, \neg}, F, AX = \{A_1, A_2, A_3\}, MP) \]

A1 \hspace{1em} (A \Rightarrow (B \Rightarrow A)),

A2 \hspace{1em} ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),

A3 \hspace{1em} (((A \Rightarrow B) \Rightarrow A) \Rightarrow A)

MP \hspace{1em} (Rule of inference)

\[ (MP) \frac{A; (A \Rightarrow B)}{B} \]

Solution: A1, A2 are sound and MP rule is sound, as proved for the system $H_1$. We have to prove that A3 is sound, i.e.

\[ \models (((A \Rightarrow B) \Rightarrow A) \Rightarrow A). \]

Assume that \(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)\), we get $A = F$ and $((A \Rightarrow B) \Rightarrow A) = F$. We evaluate $(F \Rightarrow B) = T$ if and only if $(F \Rightarrow B) = F$, what is impossible.

(2) Does Deduction Theorem holds for $H$? Justify your answer.

Solution: Yes, it does. Our system $H$ contains all axioms of $H_1$ and these were the only axioms needed for the proof of Deduction Theorem.

(3) Is $H$ COMPLETE with respect to all classical semantics tautologies? Justify your answer.

Solution: The system $H$ is not complete under classical semantics. It means that not all classical tautologies have a proof in $H$. We have proved that one needs negation and one of other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can’t express negation in terms of implication and hence we can’t provide a proof of any tautology i.e. its logically equivalent form in our language.

(4) Verify whether $H$ is SOUND under $L$ semantics.
**Solution:** $H$ is NOT sound under $L$ semantics. The axiom $A3$ is not sound. Any truth assignment such that $A = \bot, B = F$ is a counter model for $A3$.

We evaluate:

\[
((A \Rightarrow B) \Rightarrow (A \Rightarrow B)) = \bot \quad \text{if and only if} \quad ((A \Rightarrow B) \Rightarrow A) = T \quad \text{and} \quad A = \bot.
\]

Consequently, $((\bot \Rightarrow \bot) \Rightarrow \bot) = T$ for $B = F$.

**QUESTION 6**

$S$ is the following (sound) proof system:

\[
S = ( L_{\{\Rightarrow, \cap\}}, \mathcal{F}, AX = \{A1\} \quad R = \{(r1), (r2)\}),
\]

where

**Axiom:** $A1 = (B \Rightarrow (A \Rightarrow B))$,

**Rules:**

\[
\begin{align*}
(r1) & \quad \frac{A ; B}{(A \cap B)} \\
(r2) & \quad \frac{A ; (C \cap D)}{(A \Rightarrow (C \cap D))}
\end{align*}
\]

For the sequence $B1, B2, B3, B4$ of formulas of $L_{\{\Rightarrow, \cap\}}$ defined below determine if $B1, B2, B3, B4$ form a FORMAL PROOF in $S$.

If YES, provide comments how each step of the proof was obtained. Write your comments in the space between the steps.

If NOT, write the reason in a proper space between the steps.

**Solution:**

\[
\begin{align*}
B1 &= (A \Rightarrow (B \Rightarrow A)), \\
\text{Axiom A1} \\
B2 &= (B \Rightarrow (A \Rightarrow B)), \\
\text{Axiom A1 for } A = B, B = A \\
B3 &= ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A))), \\
B1, B2 \text{ and } (r1) \text{ for } A = B_2, B = B_1 \\
B4 &= ((A \Rightarrow (B \Rightarrow A)) \Rightarrow ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A)))) \\
B1, B4 \text{ and } (r2) \text{ for } A = B_2, (C \cap D) = B_4, \\
i.e. \ C = ((B \Rightarrow (A \Rightarrow B)), D = (A \Rightarrow (B \Rightarrow A)).
\end{align*}
\]

**QUESTION 7**

Let $H$ be the proof system defined in QUESTION 5.

(a) Prove the following: $A \vdash_H (A \Rightarrow A)$

**Solution 1:** Proof is as follows.

\[
\begin{align*}
B1 &= A \Rightarrow (A \Rightarrow A) \\
\text{Axiom A1 for } B = A \\
B2 &= A \\
\text{Hypothesis}
\end{align*}
\]
$B_3 = (A \Rightarrow A)$

$B_1, B_2$ and MP

**Solution 2:** We use Deduction Theorem.

$A \vdash_H (A \Rightarrow A)$ if and only if $\vdash_H (A \Rightarrow (A \Rightarrow A))$, what is true because $(A \Rightarrow (A \Rightarrow A))$ is axiom $A_1$. The proof is one element sequence:

$B_1 = A \Rightarrow (A \Rightarrow A)$

Axiom $A_1$ for $B = A$

**(b)** We know that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_H B$.

**Solution 1:** We apply Deduction Theorem twice:

$\vdash_H (\neg A \Rightarrow (A \Rightarrow B)$ if and only if $\neg A \vdash_H (A \Rightarrow B)$ if and only if $\neg A, A \vdash_H B$.

**Solution 2:** We construct the formal proof of $\neg A, A \vdash_H B$ as follows.

$B_1 = (\neg A \Rightarrow (A \Rightarrow B))$

Assumption that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B)$

$B_2 = \neg A$

Hypothesis

$B_3 = A$

Hypothesis

$B_4 = (A \Rightarrow B)$

$B_1, B_2$ and MP

$B_5 = B$

$B_3, B_4$ and MP