SOME BASIC DEFINITIONS 2

ORDER RELATIONS, LATTICES, BOOLEAN ALGEBRAS

Order Relation $R \subset A \times A$ is an order on A iff R is 1.Reflexive, 2. Antisymmetric, 3. Transitive, i.e.

1. $\forall a \in A\ (a, a) \in R$
2. $\forall a, b \in A\ ((a, b) \in R \cap (b, a) \in R\ \Rightarrow\ a = b)$
3. $\forall a, b, c \in A\ ((a, b) \in R \cap (b, c) \in R\ \Rightarrow\ (a, c) \in R)$

Total Order $R \subset (A \times A)$ is a total order on A iff R is an order and any two elements of A are comparable, i.e.
$\forall a, b \in A\ ((a, b) \in R \cup (b, a) \in R)$.

Historical names Order is also called **partial order** and total order is also called a **linear order**.

Notations Order relations are usually denoted by $\leq$. We use, in our lecture notes the notation $\boxed{\leq}$. Our book, and hence this handout uses the notation $\preceq$ as a symbol for order relation. Remember, that even if we use $\leq$ as the order relation symbol, it is a SYMBOL for ANY order relation and not only a symbol for a natural order $\leq$ in number sets.

Poset A set $A \neq \emptyset$ ordered by a relation R is called a poset. We write it as a tuple: (A, R), $(A, \leq)$, $(A, \preceq)$ or (A, $\boxed{\leq}$). Name poset stands for "partially ordered set".

Diagram Diagram or Hasse Diagram of order relation is a graphical representation of a poset. It is a simplified graph constructed as follows.

1. As the relation is REFLEXIVE, i.e. $(a, a) \in R$ for all $a \in A$, we draw a point a instead of a point a with the loop.

2. As the relation is antisymmetric we draw a point b **above** point a (connected, but without the arrow) to indicate that $(a, b) \in R$.

3. As the relation in transitive, we connect points a, b, c without arrows.

Special elements in a poset $(A, \preceq)$ (book notation) are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least) $a_0 \in A$ is a smallest (least) element in the poset $(A, \preceq)$ iff $\forall a \in A\ (a_0 \preceq a)$.

Greatest (largest) $a_0 \in A$ is a greatest (largest) element in the poset $(A, \preceq)$ iff $\forall a \in A\ (a \preceq a_0)$.

Maximal (formal) $a_0 \in A$ is a maximal element in the poset $(A, \preceq)$ iff $\neg\ \exists a \in A\ (a_0 \preceq a\ \cap\ a_0 \neq a)$.

Maximal (informal) $a_0 \in A$ is a maximal element in the poset $(A, \preceq)$ iff on the diagram of $(A, \preceq)$ there is no element placed above a_0.

Minimal $a_0 \in A$ is a minimal element in the poset $(A, \preceq)$ iff $\neg\ \exists a \in A\ (a \preceq a_0\ \cap\ a_0 \neq a)$.

Minimal (informal) $a_0 \in A$ is a minimal element in the poset $(A, \preceq)$ iff on the diagram of $(A, \preceq)$ there is no element placed below a_0.

Lower Bound Let $B \subseteq A$ and $(A, \preceq)$ is a poset. $a_0 \in A$ is a lower bound of a set B iff $\forall b \in B\ (a_0 \preceq b)$.

Upper Bound Let $B \subseteq A$ and $(A, \preceq)$ is a poset. $a_0 \in A$ is an upper bound of a set B iff $\forall b \in B\ (b \preceq a_0)$.

Least upper bound of B (lub B) Given: a set $B \subseteq A$ and $(A, \preceq)$ a poset. $x_0 = lubB$ iff x_0 is (if exists) the least (smallest) element in the set of all upper bounds of B, ordered by the poset order $\preceq$.

Greatest lower bound of B (glb B) Given: a set $B \subseteq A$ and $(A, \preceq)$ a poset. $x_0 = glbB$ iff x_0 is (if exists) the greatest element in the set of all lower bounds of B, ordered by the poset order $\preceq$.

Lattice A poset $(A, \preceq)$ is a lattice iff For all $a, b \in A$ both $lub\{a, b\}$ and $glb\{a, b\}$ exist.

Lattice notation Observe that by definition elements $lubB$ and $glbB$ are always unique (if they exist). For $B = \{a, b\}$ we denote:
$lub\{a, b\} = a \cup b$ and $glb\{a, b\} = a \cap b$.

Lattice union (meet) The element $lub\{a, b\} = a \cup b$ is called a lattice union (meet) of a and b. By lattice definition for any $a, b \in A$ $a \cup b$ always exists.

Lattice intersection (joint) The element $glb\{a, b\} = a \cap b$ is called a lattice intersection (joint) of a and b. By lattice definition for any $a, b \in A$ $a \cap b$ always exists.

Lattice as an Algebra An algebra $(A, \cup, \cap)$, where $\cup, \cap$ are two argument operations on A is called a lattice iff the following conditions hold for any $a, b, c \in A$ (they are called lattice AXIOMS):

- **l1** $a \cup b = b \cup a$ and $a \cap b = b \cap a$
- **l2** $(a \cup b) \cup c = a \cup (b \cup c)$ $and\$(a \cap b) \cap c = a \cap (b \cap c)$
- **l3** $a \cap (a \cup b) = a$ and $a \cup (a \cap b) = a$.

Lattice axioms The conditions **l1- l3** from above definition are called lattice axioms.

Lattice orderings Let the $(A, \cup, \cap)$ be a lattice. The relations:

$a \preceq b$ iff $a \cup b = b$, $a \preceq b$ iff $a \cap b = a$
are order relations in A and are called a lattice orderings.

Distributive lattice A lattice $(A, \cup, \cap)$ is called a distributive lattice iff for all $a, b, c \in A$ the following conditions hold

- **l4** $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$
- **l5** $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$.

Distributive lattice axioms Conditions l1- l5 from above are called a distributive lattice axioms.

Lattice special elements The greatest element in a lattice (if exists) is denoted by 1 and is called a lattice UNIT. The least (smallest) element in A (if exists)is denoted by 0 and called a lattice zero.

Lattice with unit and zero If 0 (lattice zero) and 1 (lattice unit) exist in a lattice, we will write the lattice as: $(A, \cup, \cap, 0, 1)$ and call is a lattice with zero and unit.

Lattice Unit Axioms Let $(A, \cup, \cap)$ be a lattice. An element $x \in A$ is called a lattice unit iff for any $a \in A$ $x \cap a = a$ and $x \cup a = x$.
If such element x exists we denote it by 1 and we write the unit axioms as follows.

- **l6** $1 \cap a = a$
- **l7** $1 \cup a = 1$.

Lattice Zero Axioms Let $(A, \cup, \cap)$ be a lattice. An element $x \in A$ is called a lattice zero iff for any $a \in A$ $x \cap a = x$ and $x \cup a = a$.
We denote the lattice zero by 0 and write the zero axioms as follows.

l8 $0 \cap a = 0$

l9 $0 \cup a = a.$

Complement Let $(A, \cup, \cap, 1.0)$ be a lattice with unit and zero. An element $x \in A$ is called a complement of an element $a \in A$ iff $a \cup x = 1$ and $a \cap x = 0$.

Complement axioms Let $(A, \cup, \cap, 1.0)$ be a lattice with unit and zero. The complement of $a \in A$ is usually denoted by $-a$ and the above conditions that define the complementabove are called complement axioms. The complement axioms are usually written as follows.

c1 $a \cup -a = 1$

c2 $a \cap -a = 0.$

Boolean Algebra A distributive lattice with zero and unit such that each element has a complement is called a Boolean Algebra.

Boolean Algebra Axioms A lattice $(A, \cup, \cap, 1.0)$ is called a Boolean Algebra iff the operations $\cap, \cup$ satisfy axioms **l1 -l5**, $0 \in A$and $1 \in A$ satisfy axioms **l6 - l9** and each element $a \in A$ has a complement $-a \in A$, i.e.

l1o $\forall a \in A\ \exists - a \in A\ ((a \cup -a = 1)\ \cap\ (a \cap -a = 0)).$

SOME BASIC FACTS

Uniqueness In any poset $(A, \preceq)$, if a greatest and a least elements exist, then they are unique.

Finite Posets If $(A, \preceq)$ is a finite poset (i.e. A is a finite set), then a unique maximal (if exists) is the largest element and a unique minimal (if exists) is the least element.

Infinite Posets It is possible to to order an infinite set A in such a way that the poset $(A, \preceq)$ has a unique maximal element (minimal element) and no largest element (least element).

Any poset In any poset, the largest element is a unique maximal element and the least element is the unique minimal element.

Lower, upper bounds A set $B \subseteq A$ of a poset $(A, \preceq)$ can have none, finite or infinite number of lower or upper bounds, depending of ordering.

Finite lattice If $(A, \cup, \cap)$ is a finite lattice (i.e. A is a finite set), then 1 and 0 always exist.

Infinite lattice If $(A, \cup, \cap)$ is an infinite lattice (i.e. the set A is infinite), then 1 or 0 might or might not exist.

For example:

$(N \leq)$ is a lattice with 0 (the number 0) and no 1.

$(Z \leq)$ is a lattice without 0 and without 1.

Finite Boolean Algebra Non- generate Finite Boolean Algebras always have 2^n elements ($n \geq 1$.

Representation Theorem any Boolean algebra is isomorphic with the Boolean algebra of sets.