1 Completeness Theorem for First Order Logic

There are many proofs of the Completeness Theorem for First Order Logic. We follow here a version of Henkin’s proof, as presented in the *Handbook of Mathematical Logic*. It contains a method for reducing certain problems of first-order logic back to problems about propositional logic. We give independent proof of Compactness Theorem for propositional logic. The Compactness Theorem for first-order logic and Löwenheim-Skolem Theorems and the Gödel Completeness Theorem fall out of the Henkin method.

1.1 Compactness Theorem for Propositional Logic

Let $\mathcal{L} = \mathcal{L}(P, F, C)$ be a first order language with equality. We assume that the sets $P, F, C$ are infinitely enumerable. We define a propositional logic within it as follows.

**Prime formulas** We consider a subset $P$ of the set $F$ of all formulas of $\mathcal{L}$. Intuitively these are formulas of $\mathcal{L}$ which are not direct propositional combination of simpler formulas, that is, *atomic formulas* ($A \setminus F$) and formulas beginning with quantifiers.

Formally, we have that $P = \{ A \in F : A \in A \setminus F \text{ or } A = \forall x B, A = \exists x B \text{ for } B \in F \}$.

**Example 1.1** The following are primitive formulas.

$R(t_1, t_2), \forall x (A(x) \Rightarrow \neg A(x)), (c = c), \exists x (Q(x, y) \cap \forall y A(y))$.

The following are not primitive formulas.

$(R(t_1, t_2) \Rightarrow (c = c)), (R(t_1, t_2) \cup \forall x (A(x) \Rightarrow \neg A(x)))$.

Given a set $P$ of primitive formulas we define in a standard way the set $P \mathcal{F}$ of propositional formulas as follows.

**Propositional formulas** The smallest set $P \mathcal{F} \subset \mathcal{F}$ such that

1. $P \subset P \mathcal{F}$
2. If $A, B \in P \mathcal{F}$, then $(A \Rightarrow B), (A \cup B), (A \cap B)$, and $\neg A \in P \mathcal{F}$

is called a set of propositional formulas of the first order language $\mathcal{L}$.

We define propositional semantics for propositional formulas in $P \mathcal{F}$ as follows.
Truth assignment  Let $P$ be a set of prime formulas and $\{T, F\}$ be a two element set, thought as the set of logical values "true" and "false". Any function

$$v : P \rightarrow \{T, F\}$$

is called truth assignment (or variable assignment).

Let $B = (\{T, F\}, \Rightarrow, \cup, \cap, \neg)$ be a two-element Boolean algebra and $PF = (PF, \Rightarrow, \cup, \cap, \neg)$ a similar algebra of propositional formulas.

We extend $v$ to a homomorphism

$$v^* : PF \rightarrow B$$
in a usual way, i.e. we put $v^*(A) = v(A)$ for $A \in P$, and for any $A, B \in PF$,

$$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B),$$

$$v^*(A \cup B) = v^*(A) \cup v^*(B),$$

$$v^*(A \cap B) = v^*(A) \cap v^*(B),$$

$$v^*(\neg A) = \neg v^*(A).$$

Propositional Model  A truth assignment $v$ is called a propositional model for a formula $A \in PF$ iff $v^*(A) = T$.

Propositional Tautology  A formula $A \in PF$ is a propositional tautology if $v^*(A) = T$ for all $v : P \rightarrow \{T, F\}$.

For the sake of simplicity we will often say model, tautology instead propositional model, propositional tautology.

Model for the Set  Given a set $S$ of propositional formulas. We say that $v$ is a model for the set $S$ if $v$ is a model for all formulas $A \in S$.

Consistent Set  A set $S$ of propositional formulas is consistent (in a sense of propositional logic) if it has a (propositional) model.

Theorem 1.1 (Compactness Theorem for Propositional Logic) A set $S$ of propositional formulas is consistent if and only if every finite subset of $S$ is consistent.

**proof**  If $S$ is a consistent set, then its model is also a model for all its finite subsets and all its finite subsets are consistent.

We prove the nontrivial half of the Compactness Theorem in a slightly modified form. To do so, we introduce the following definition.
Finitely Consistent Set (FC) Any set $S$ such that all its subsets are consistent is called finitely consistent.

We use this definition to re-write the Compactness Theorem as: A set $S$ of propositional formulas is consistent if and only if it is finitely consistent. The nontrivial half of it is:

*Every finitely consistent set of propositional formulas is consistent.*

The proof of the nontrivial half of the Compactness Theorem, as stated above, consists of the following four steps.

**Step 1** We introduce the notion of a *maximal finitely consistent set*.

**Step 2** We show that every maximal finitely consistent set is consistent by constructing its model.

**Step 3** We show that every finitely consistent set $S$ can be extended to a maximal finitely consistent set $S^*$. I.e we show that for every finitely consistent set $S$ there is a set $S^*$, such that $S \subseteq S^*$ and $S^*$ is maximal finitely consistent.

**Step 4** We use steps 2 and 3 to justify the following reasoning. Given a finitely consistent set $S$. We extend it, via construction defined in the step 2 to a maximal finitely consistent set $S^*$. By the step 2, $S^*$ is consistent and hence so is the set $S$, what ends the proof.

**Step 1: Maximal Finitely Consistent Set** We call $S$ maximal finitely consistent if $S$ is finitely consistent and for every formula $A$, either $A \in S$.

We use notation MFC for maximal finitely consistent set, and FC for the finitely consistent set.

**Step 2: Any MFC set is consistent** Given a MFC set $S^*$, we prove its consistency by constructing a truth assignment $v : P \rightarrow \{T, F\}$ such that for all $A \in S^*$, $v^*(A) = T$.

Observe that the MFC sets have the following property.

**MCF Property** For any MFC set $S^*$, for every $A \in PF$, exactly one of the formulas $A \land \neg A$ belongs to $S^*$.

In particular, for any $P \in PF$, we have that exactly one of $P, \neg P \in S^*$. This justify the correctness of the following definition.

Let $v : P \rightarrow \{T, F\}$ be a mapping such that
\[ v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases} \]

We extend \( v \) to \( v^* : \mathcal{P} \mathcal{F} \rightarrow \mathcal{B} \) in a usual way. In order to prove that \( v \) is a model for \( S^* \) we have to show that for any \( A \in \mathcal{P} \mathcal{F} \),

\[ v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases} \]

We prove it by induction on the degree of the formula \( A \). The base case of \( A \in \mathcal{P} \mathcal{F} \) follows immediately from the definition of \( v \).

**Case** \( A = \neg C \)  
Assume that \( A \in S^* \). This means \( \neg C \in S^* \) and by MCF Property we have that \( C \notin S^* \). So by the inductive assumption \( v^*(C) = F \) and \( v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T \).

Assume now that \( A \notin S^* \). By MCF Property we have that \( C \in S^* \). By the inductive assumption \( v^*(C) = T \) and \( v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F \).

This proves that for any formula \( A \),

\[ v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases} \]

**Case** \( A = (B \cup C) \)  
Let \( (B \cup C) \in S^* \). It is enough to prove that in this case \( B \in S^* \) and \( C \in S^* \), because then from the inductive assumption \( v^*(C) = v^*(D) = T \) and \( v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup T = T \).

Assume that \((B \cup C) \in S^* \), \( B \notin S^* \) and \( C \notin S^* \). Then by MCF Property we have that \( \neg B \in S^* \), \( \neg C \in S^* \) and consequently the set

\[ \{(B \cup C), \neg B, \neg C\} \]

is a finite inconsistent subset of \( S^* \), what contradicts the fact that \( S^* \) is finitely consistent.

Assume now that \( (B \cup C) \notin S^* \). By MCF Property, \( \neg(B \cup C) \in S^* \) and by the \( A = \neg C \) we have that \( v^*(\neg(B \cup C)) = T \). But \( v^*(\neg(B \cup C)) = \neg v^*(B \cup C) = T \) means that \( v^*((B \cup C)) = F \), what end the proof of this case.

The remaining cases of \( A = (B \cap C) \), \( A = (B \Rightarrow C) \) are similar to the above and are left to the reader as an exercise.
Step 3: Maximal finitely consistent extension  

Given a finitely consistent set $S$, we construct its \textit{maximal finitely consistent extension} $S^*$ as follows.

The set of all formulas of $\mathcal{L}$ is countable, so is $PF$. We assume that all propositional formulas form a one-to-one sequence

$$A_1, A_2, ...., A_n, ....$$

(1)

We define a chain

$$S_0 \subset S_1 \subset S_2 \ldots \subset S_n \subset ....$$

(2)

of \textit{extensions} of the set $S$ by

$$S_0 = S;$$

$$S_{n+1} = \begin{cases} 
S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is finitely consistent} \\
S_n \cup \{\neg A_n\} & \text{otherwise.} 
\end{cases}$$

We take

$$S^* = \bigcup_{n \in \mathbb{N}} S_n.$$  

(3)

Clearly, $S \subset S^*$ and for every $A$, either $A \in S^*$ or $\neg A \in S^*$. To finish the proof that $S^*$ is MCF we have to show that it is finitely consistent.

First, let observe that if all sets $S_n$ are finitely consistent, so is $S^* = \bigcup_{n \in \mathbb{N}} S_n$. Namely, let $S_F = \{B_1, ..., B_k\}$ be a finite subset of $S^*$. This means that there are sets $S_{i_1}, ..., S_{i_k}$ in the chain (2) such that $B_m \in S_{i_m}$, $m = 1, .. k$. Let $M = \text{max}(i_1, ..., i_k)$. Obviously $S_F \subset S_M$ and $S_M$ is finitely consistent as an element of the chain (2). This proves the if all sets $S_n$ are finitely consistent, so is $S^*$.

Now we have to prove only that all $S_n$ in the chain (2) are finitely consistent. We carry the proof by induction over the length of the chain. $S_0 = S$, so it is FC by assumption of the Compactness Theorem. Assume now that $S_n$ is FC, we prove that so is $S_{n+1}$. We have two cases to consider.

\textbf{Case 1}  

$S_{n+1} = S_n \cup \{A_n\}$, then $S_{n+1}$ is FC by the definition of the chain (2).
**Case 2**  \( S_{n+1} = S_n \cup \{\neg A_n\} \). Observe that this can happen only if \( S_n \cup \{A_n\} \) is not FC, i.e. there is a finite subset \( S'_n \subset S_n \), such that \( S'_n \cup \{A_n\} \) is not consistent.

Suppose now that \( S_{n+1} \) is not FC. This means that there is a finite subset \( S''_n \subset S_n \), such that \( S''_n \cup \{\neg A_n\} \) is not consistent.

Take \( S'_n \cup S''_n \). It is a finite subset of \( S_n \) so is consistent by the inductive assumption. Let \( v \) be a model of \( S'_n \cup S''_n \). Then one of \( v^*(A), v^*(\neg A) \) must be \( T \). This contradicts the inconsistency of both \( S'_n \cup \{A_n\} \) and \( S'_n \cup \{\neg A_n\} \).

Thus, in either case, \( S_{n+1} \), is after all consistent. This ends the proof of the Step 3 and of the Compactness Theorem via the argument presented in the Step 4.

### 1.2 Reduction of first-order logic to propositional logic

Propositional tautologies as defined in the previous section barely scratch the surface of the collection of first-order tautologies, or first order *valid* formulas, as they are often called. For example the following first-order formulas are propositional tautologies,

\[
(\exists x A(x) \cup \neg \exists x A(x)),
\]

\[
(\forall x A(x) \cup \neg \forall x A(x)),
\]

\[
(\neg (\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x))),
\]

but the following are first order tautologies (valid formulas) that are not propositional tautologies:

\[
\forall x (A(x) \cup \neg A(x)),
\]

\[
(\neg \forall x A(x) \Rightarrow \exists x \neg A(x)).
\]

The first formula above is just a prime formula, the second is of the form \((\neg B \Rightarrow C)\), for \( B \) and \( C \) prime.

To stress the difference between the propositional and first order tautologies some books reserve the word *tautology* for the propositional tautologies alone, using the notion of *valid formula* for the first order tautologies. We use here both notions, with the preference to *first-order tautology* or *tautology* for short when there is no room for misunderstanding.

To make sure that there is no misunderstandings we remind the following definitions.
Given a first order language $\mathcal{L}$ with the set of variables $VAR$ and the set of formulas $\mathcal{F}$. Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$, with the universe $M$ and the interpretation $I$ and let $s : VAR \rightarrow M$ be a valuation of $\mathcal{L}$ in $M$.

**A is true in $\mathcal{M}$** Given a structure $\mathcal{M} = [M, I]$, we say that a formula $A$ is true in $\mathcal{M}$ if there is a valuation $s : VAR \rightarrow M$ such that

$$(\mathcal{M}, s) \models A.$$

**A is valid in $\mathcal{M}$** Given a structure $\mathcal{M} = [M, I]$, we say that a formula $A$ is valid in $\mathcal{M}$ if

$$(\mathcal{M}, s) \models A$$

for all valuations $s : VAR \rightarrow M$.

**Model $\mathcal{M}$** If $A$ is valid in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a model of $A$.

**A is valid** A formula $A$ called is valid if it is valid in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are models of $A$.

**A is a first-order tautology** A valid formula $A$ is also called a first-order tautology, or tautology, for short.

**Case: $A$ is a sentence** If $A$ is a sentence, then the truth or falsity of $$(\mathcal{M}, s) \models A$$ is completely independent of $s$. Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ is a model of $A$, if for some (hence every) valuation $s$, $$(\mathcal{M}, s) \models A.$$

**Model of a set of sentences** $\mathcal{M}$ is a model of a set $S$ of sentences if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S.$$