QUESTION 1

\( H \) is the following proof system:

\[
H = ( \mathcal{L}_{\{\Rightarrow, \neg}\}, \mathcal{F}, \ AX = \{A1, A2, A3, A4\}, \ MP )
\]

A1 \((A \Rightarrow (B \Rightarrow A)),\)

A2 \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))\),

A3 \(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))\)

A4 \(((A \Rightarrow B) \Rightarrow A) \Rightarrow A\)

MP (Rule of inference)

\[
(MP) \frac{A; (A \Rightarrow B)}{B}
\]

(1) Prove that \( H \) is SOUND under classical semantics.

(2) Does Deduction Theorem holds for \( H \)? Justify shortly your answer.

(3) Is \( H \) COMPLETE with respect to all classical semantics tautologies? JUSTIFY your answer.
QUESTION 2  Let $H$ be the proof system defined in QUESTION 1.

(a)  Prove the following:  $A \vdash_H (A \Rightarrow A)$

(b) We know that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_H B$.

QUESTION 3  Here are consecutive steps $B_1, ..., B_5$ in the formal proof in $H_2$ of

$(B \Rightarrow \neg\neg B)$

$B_1 = ((\neg\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
\[ B_3 = ((\neg\neg B \Rightarrow B) \Rightarrow \neg B) \]

\[ B_4 = (B \Rightarrow (\neg\neg B \Rightarrow B)) \]

\[ B_5 = (B \Rightarrow \neg\neg B) \]

**Complete** the steps \[ B_1, \ldots, B_5 \]

of the proof by writing all details in the space provided below each step of the proof.

You have to write down **the proper substitutions and formulas** used at each step of the proof.

**You can use** the following already proved facts:

1. \((A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C),\)

2. \(\vdash_{H_2} (\neg B \Rightarrow B).\)
QUESTION 4 Let $A$ be a formula
\[ ((-a \Rightarrow -b) \Rightarrow c) \]
and let $v$ be such that
\[ v(a) = T, \quad v(b) = F, v(c) = F. \]
Evaluate $A', B_1, \ldots, B_n$ as defined by the following definition.

**Definition** Let $A$ be a formula and $b_1, b_2, \ldots, b_n$ be all propositional variables that occur in $A$.
Let $v$ be variable assignment $v : VAR \rightarrow \{T, F\}$.

We define, for $A, b_1, b_2, \ldots, b_n$ and $v$ a corresponding formulas $A', B_1, B_2, \ldots, B_n$ as follows: (for $i = 1, 2, \ldots, n$)
\[
A' = \begin{cases} 
A & \text{if } v^*(A) = T \\
\neg A & \text{if } v^*(A) = F 
\end{cases}
\]
\[
B_i = \begin{cases} 
b_i & \text{if } v(b_i) = T \\
\neg b_i & \text{if } v(b_i) = F 
\end{cases}
\]

QUESTION 5 Consider a system $RS_2$ obtained from $RS$ by changing the sequence $\Gamma'$ into $\Gamma$ and $\Delta$ into $\Delta'$ in all of the rules of inference of $RS$.

1. Construct a decomposition tree in $RS_2$ of \[((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))\]
2. Define in your own words, for any $A$, the decomposition tree $T_A$ in $RS2$. 