QUESTION 1 Use the fact that \( v : \text{VAR} \longrightarrow \{ F, \bot, T \} \) be such that \( v^*(\langle a \land b \Rightarrow \neg b \rangle) = \bot \) under \( \mathbb{L} \) semantics to evaluate \( v^*(\langle (b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b) \rangle \cup (a \Rightarrow b)) \). Use shorthand notation.

\[
\begin{array}{c|ccc}
\neg & F & \bot & T \\
\hline
T & \bot & F \\
\end{array}
\quad
\begin{array}{c|ccc}
\cup & F & \bot & T \\
\hline
F & F & F & T \\
\bot & \bot & \bot & T \\
T & T & T & T \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\land & F & \bot & T \\
\hline
F & F & F & F \\
\bot & \bot & \bot & T \\
T & F & T & T \\
\end{array}
\quad
\begin{array}{c|ccc}
\Rightarrow & F & \bot & T \\
\hline
F & T & T & T \\
\bot & \bot & T & T \\
T & F & T & T \\
\end{array}
\]

Solution: \( \langle (a \land b) \Rightarrow \neg b \rangle = \bot \) in two cases.

C1 \( (a \land b) = \bot \) and \( \neg b = F \).

C2 \( (a \land b) = T \) and \( \neg b = \bot \).

Case C1: \( \neg b = F \), i.e. \( b = T \), and hence \( (a \land T) = \bot \) iff \( a = \bot \). We get that \( v \) is such that \( v(a) = \bot \) and \( v(b) = T \).

We evaluate: \( v^*(\langle (b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b) \rangle \cup (a \Rightarrow b)) = \langle (T \Rightarrow \bot) \Rightarrow (\bot \Rightarrow \neg T) \rangle \cup (\bot \Rightarrow T) \rangle = \langle (\bot \Rightarrow \bot) \cup T \rangle = T \).

Case C2: \( \neg b = \bot \), i.e. \( b = \bot \), and hence \( (a \land \bot) = T \) what is impossible, hence \( v \) from case C1 is the only one.

QUESTION 2 Prove using proper logical equivalences (list them at each step) that
\[
\neg((A \Rightarrow \neg B) \cup (B \Rightarrow \neg A)) \equiv (A \land B).
\]

Solution: \( \neg((A \Rightarrow \neg B) \cup (B \Rightarrow \neg A)) \equiv_{\text{deMorgan}} (\neg(a \Rightarrow b) \cap (b \Rightarrow \neg A)) \equiv_{\text{impl}} ((A \lor \neg B) \cap (B \lor \neg A)) \equiv_{\text{assoc,comm}} (A \land B) \).

QUESTION 3 Given a proof system:
\[
S = \langle \mathcal{L}_{\cup,\Rightarrow} \rangle, \quad \mathcal{E} = \mathcal{F} \quad AX = \{ A_1, A_2 \}, \quad \mathcal{R} = \{ (r) \},
\]

where

\[ A_1 = (A \Rightarrow (A \cup B)), \quad A_2 = (A \Rightarrow (B \Rightarrow A)) \]

and

\[
(r) \quad \frac{(A \Rightarrow B)}{(A \Rightarrow (A \Rightarrow B))}
\]

1. **Solution:** Prove that \( S \) is *sound* under classical semantics.

**Solution:** Axioms of \( S \) are basic classical tautologies. The proof of soundness of the rule of inference is the following.

Assume \((A \Rightarrow B) = T\). Hence the logical value of conclusion is \((A \Rightarrow (A \Rightarrow B)) = (A \Rightarrow T) = T\) for all \(A\).

2. Determine whether \( S \) is *sound* or *not sound* under \( K \) semantics.

**K semantics** differ from Łukasiewicz’s semantics only in a case on implication only. This table is:

<table>
<thead>
<tr>
<th>(\Rightarrow)</th>
<th>(F)</th>
<th>(\bot)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(\bot)</td>
<td>(\bot)</td>
<td>(\bot)</td>
<td>(T)</td>
</tr>
<tr>
<td>(T)</td>
<td>(F)</td>
<td>(\bot)</td>
<td>(T)</td>
</tr>
</tbody>
</table>

**Solution 1:** \( S \) is not sound under \( K \) semantics. Let’s take truth assignment such that \( A = \bot, B = \bot \). The logical value of axiom \( A_1 \) is as follows.

\[(A \Rightarrow (A \cup B)) = (\bot \Rightarrow (\bot \cup \bot)) = \bot \text{ and } \not\models_K (A \Rightarrow (A \cup B)).\]

**Observe** that the \(v\) such that \( A = \bot, B = \bot \) is not the only \(v\) that makes \( A_1 \not\models T\), i.e. proves that \( \not\models_K A_1 \).

\((A \Rightarrow (A \cup B)) \neq T\) if and only if \((A \Rightarrow (A \cup B)) = F\) or \((A \Rightarrow (A \cup B)) = \bot\). The first case is impossible because \( A_1 \) is a classical tautology.

Consider the second case. \((A \Rightarrow (A \cup B)) = \bot\) in two cases.

- **c1** \( A = \bot \) and \((A \cup B) = F\), i.e. \((\bot \cup B) = F\), what is impossible.
- **c2** \( A = T \) and \((A \cup B) = \bot\), i.e. \((T \cup B) = \bot\), what is impossible.
- **c3** \( A = \bot \) and \((A \cup B) = \bot\), i.e. \((\bot \cup B) = \bot\). This is possible for \( B = \bot \) or \( B = F\), i.e when \( A = \bot, B = \bot \) or \( A = \bot, B = F\).

From the above Observation we get second solution.

**Solution 2:** \( S \) is not sound under \( K \) semantics. Axiom \( A_1 \) is not \( K \) semantics tautology. There are exactly two truth assignments \( v \), such that \( v \not\models A_1 \). One is, as defined in Solution 1: \( A = \bot, B = \bot \). The second is \( A = \bot, B = F\).

**QUESTION 4**

1. Write a formal proof \( A_1, A_2, A_3 \) in \( S \) from the QUESTION 3 with 2 applications of the rule \( r \) that starts with axiom \( A_1 \), i.e. such that \( A_1 = A_1 \).
Solution: The formal proof $A_1, A_2, A_3$ is as follows.

$A_1 = (A \Rightarrow (A \cup B))$

Axiom

$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow B)))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow B)$

$A_3 = (A \Rightarrow (A \Rightarrow (A \Rightarrow B)))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \Rightarrow B))$.

2. Use results from QUESTION 3 to determine whether $\models_K A_3$.

Solution 1: We use the two $v$ from QUESTION 3 to evaluate the logical value of $A_3$. Namely we evaluate:

$v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow B)))) = (\bot \Rightarrow (\bot \Rightarrow (\bot \Rightarrow (\bot \Rightarrow \bot)))) = \bot$, or $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow B)))) = (\bot \Rightarrow (\bot \Rightarrow (\bot \Rightarrow (\bot \Rightarrow \bot)))) = \bot$. Both cases prove that $\not\models_K A_3$.

Solution 2: We know that $S$ is not sound, because there is $v$ for which $A_1 = A_1 = \bot$, as evaluated in Question 3. We prove that the rule (r) preserves the logical value $\bot$ under any $v$ such that $A_1 = \bot$. as follows.

Let the premiss $(A \Rightarrow B) = \bot$, the logical value of the conclusion is hence $(A \Rightarrow \bot) = \bot$ for $A = \bot, T$ and $(A \Rightarrow \bot) = T$ for $A = F$.

The case $A = F$ evaluates the premiss $(A \Rightarrow B) = (F \Rightarrow B) = T$ for all B, what contradicts the assumption that $(A \Rightarrow B) = \bot$. We must hence have $A = \bot$. But all possible $v$ for which $A_1 = \bot$ are such that $A = \bot$, what end the proof.

It means that any $A$ such that $A$ has proof that starts with axiom $A_1$ and then multiple applications of the rule (r) is evaluated to $\bot$ under all $v$, such that $v^*(A_1) = \bot$. Hence, in particular, $\not\models_K A_3$.

3. Write a formal proof $A_1, A_2$ from the QUESTION 3 with 1 application of the rule (r) that starts with axiom $A_2$, i.e. such that $A_1 = A_2$.

Solution: The formal proof $A_1, A_2$ is as follows.

$A_2 = (A \Rightarrow (B \Rightarrow A))$

Axiom

$A_2 = (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$

Rule (r) application for $A = A$ and $B = (B \Rightarrow A)$.

4. Use results from QUESTION 3 to determine whether $\models A_2$.

Solution: System $S$ is sound under classical semantics, hence by the soundness theorem we get that

$\models (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$,

as it has a proof in $S$.

QUESTION 5 Prove, by constructing a formal proof in $S$ from the QUESTION 3 that

$\models_S (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$.

Solution: $A_2 = (A \Rightarrow (A \Rightarrow A))$

Axiom for $B = A$

$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow A)$.

$(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A))))$

Rule (r) application for $A = A$ and $B = (A \Rightarrow (A \Rightarrow A))$. 

3