Remainder: We define $H$ semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = \max\{a, b\}, \quad a \cap b = \min\{a, b\}.$$ 

The Truth Tables for Implication and Negation are:

**H-Implication**

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>$F$</th>
<th>$\bot$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$\bot$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

**H Negation**

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>$F$</th>
<th>$\bot$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td></td>
</tr>
</tbody>
</table>

**QUESTION 1** We know that

$$v : VAR \rightarrow \{F, \bot, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \bot$$

under $H$ semantics. **evaluate**

$$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)).$$

**QUESTION 2**

We define a 4 valued $L_4$ logic semantics as follows. The language is $L = L_{\{\neg, \Rightarrow, \cup, \cap\}}$. The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $L_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$, defined as follows.
Negation $\neg$ is a function $\neg : \{\bot_1, \bot_2, T\} \rightarrow \{\bot_1, \bot_2, T\}$, such that
$$\neg \bot_1 = \bot_1, \quad \neg \bot_2 = \bot_2, \quad \neg F = T, \quad \neg T = F.$$

Conjunction $\cap$ is a function $\cap : \{\bot_1, \bot_2, T\} \times \{\bot_1, \bot_2, T\} \rightarrow \{\bot_1, \bot_2, T\}$, such that for any $a, b \in \{\bot_1, \bot_2, T\}$,
$$a \cap b = \min\{a, b\}.$$

Disjunction $\cup$ is a function $\cup : \{\bot_1, \bot_2, T\} \times \{\bot_1, \bot_2, T\} \rightarrow \{\bot_1, \bot_2, T\}$, such that for any $a, b \in \{\bot_1, \bot_2, T\}$,
$$a \cup b = \max\{a, b\}.$$

Implication $\Rightarrow$ is a function $\Rightarrow : \{\bot_1, \bot_2, T\} \times \{\bot_1, \bot_2, T\} \rightarrow \{\bot_1, \bot_2, T\}$, such that for any $a, b \in \{\bot_1, \bot_2, T\}$,
$$a \Rightarrow b = \left\{ \begin{array}{ll} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{array} \right.$$
Part 2 Verify whether 
\[ \vdash_L ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Solution:

QUESTION 3 Prove using proper logical equivalences (list them at each step) that

1. \( \neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)) \),
Solution:

2. \( ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)) \),
Solution:
QUESTION 4  We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:
\( \mathcal{L}_1 = \mathcal{L}_{\text{CON}_1} \) and \( \mathcal{L}_2 = \mathcal{L}_{\text{CON}_2} \), for \( \text{CON}_1 \neq \text{CON}_2 \).
We say that they are logically equivalent, i.e.

\[ \mathcal{L}_1 \equiv \mathcal{L}_2 \]

if and only if the following conditions C1, C2 hold.

C1:  For every formula \( A \) of \( \mathcal{L}_1 \), there is a formula \( B \) of \( \mathcal{L}_2 \), such that

\[ A \equiv B, \]

C2:  For every formula \( C \) of \( \mathcal{L}_2 \), there is a formula \( D \) of \( \mathcal{L}_1 \), such that

\[ C \equiv D. \]

Prove that \( \mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}} \).