Reminder: We define H semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = \max\{a, b\}, \quad a \cap b = \min\{a, b\}.$$ 

The Truth Tables for Implication and Negation are:

**H-Implication**

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>$\bot$</th>
<th>T</th>
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<tbody>
<tr>
<td>F</td>
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<tr>
<td>$\bot$</td>
<td>F</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>$\bot$</td>
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</tbody>
</table>

**H Negation**

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\bot$</th>
<th>T</th>
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<tbody>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
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</tbody>
</table>

**QUESTION 1 (1pt)** We know that $v : \text{VAR} \rightarrow \{ F, \bot, T \}$

is such that

$$v^*( (a \cap b) \Rightarrow (a \Rightarrow c) ) = \bot$$

under H semantics. **evaluate** $v^*((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)).$

**Solution:**
We define a 4 valued \( L_4 \) logic semantics as follows. The language is \( \mathcal{L} = \mathcal{L}(\neg, \Rightarrow, \cup, \cap) \). The logical connectives \( \neg, \Rightarrow, \cup, \cap \) of \( L_4 \) are operations in the set \( \{F, \bot_1, \bot_2, T\} \), where \( F < \bot_1 < \bot_2 < T \), defined as follows.

**Negation** \( \neg \) is a function \( \neg : \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\} \), such that

\[
\neg \bot_1 = \bot_1, \quad \neg \bot_2 = \bot_2, \quad \neg F = T, \quad \neg T = F.
\]

**Conjunction** \( \cap \) is a function \( \cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\} \), such that for any \( a, b \in \{F, \bot_1, \bot_2, T\} \),

\[
a \cap b = \min\{a, b\}.
\]

**Disjunction** \( \cup \) is a function \( \cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\} \), such that for any \( a, b \in \{F, \bot_1, \bot_2, T\} \),

\[
a \cup b = \max\{a, b\}.
\]

**Implication** \( \Rightarrow \) is a function \( \Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\} \), such that for any \( a, b \in \{F, \bot_1, \bot_2, T\} \),

\[
a \Rightarrow b = \left\{ \begin{array}{ll}
\neg a \cup b & \text{if } a > b \\
T & \text{otherwise}
\end{array} \right.
\]

**QUESTION 2** (2pts)

**Part 1** Write all TTables for \( L_4 \)

**Solution** : 

---

2
Part 2 Verify whether

\[ \models_L (\Rightarrow (a \Rightarrow b) \Rightarrow (-a \cup b)) \]

Solution:

QUESTION 3 (1pt) Prove using proper logical equivalences (list them at each step) that

1. \( \neg (A \iff B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)) \),

Solution:

2. \( ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)) \).

Solution:
QUESTION 4 (2pt) We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:
\[ L_1 = L_{\text{CON}_1} \] and \[ L_2 = L_{\text{CON}_2} \], for \( \text{CON}_1 \neq \text{CON}_2 \).

We say that they are logically equivalent, i.e.
\[ L_1 \equiv L_2 \]

if and only if the following conditions C1, C2 hold.

C1: For every formula \( A \) of \( L_1 \), there is a formula \( B \) of \( L_2 \), such that
\[ A \equiv B, \]

C2: For every formula \( C \) of \( L_2 \), there is a formula \( D \) of \( L_1 \), such that
\[ C \equiv D. \]

Prove that \( L_{\{\neg, \cap\}} \equiv L_{\{\neg, \Rightarrow\}} \).

Solution:

QUESTION 5 (2pts) Given a proof system:
\[ S = (L_{\{\neg, \Rightarrow\}}, \mathcal{F}, AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}. \]

Definition: System \( S \) is sound if and only if
(i) Axioms are tautologies and
(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that \( S \) is sound under classical semantics.
2. Prove that $S$ is not sound under $K$ semantics.

3. Write a formal proof in $S$ with 2 applications of the rule $(r)$.

QUESTION 6 (2pts)

Prove, by constructing a formal proof that

$$
\vdash_S (\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)).
$$