QUESTION 1  Prove using proper logical equivalences (list them at each step) that

1. \( \neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)) \),

Solution:

2. \( ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)) \).

Solution:

QUESTION 2  We define an EQUIVALENCE of LANGUAGES as follows:
Given two languages:
\( \mathcal{L}_1 = \mathcal{L}_{CON_1} \) and \( \mathcal{L}_2 = \mathcal{L}_{CON_2} \), for \( CON_1 \neq CON_2 \).

We say that they are logically equivalent, i.e.

\[ \mathcal{L}_1 \equiv \mathcal{L}_2 \]

if and only if the following conditions \( C_1, C_2 \) hold.

**C1:** For every formula \( A \) of \( \mathcal{L}_1 \), there is a formula \( B \) of \( \mathcal{L}_2 \), such that

\[ A \equiv B, \]

**C2:** For every formula \( C \) of \( \mathcal{L}_2 \), there is a formula \( D \) of \( \mathcal{L}_1 \), such that

\[ C \equiv D. \]

Prove that \( \mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}} \).

**QUESTION 3** Given a proof system:

\[ S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \ E = \mathcal{F}, \ AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \ (r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}. \]

**Definition:** System \( S \) is sound if and only if

(i) Axioms are tautologies and

(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that \( S \) is sound under classical semantics.
2. Prove that $S$ is not sound under $K$ semantics defined as follows.

**The language** is the same in case of classical logic.

**Connectives** $\neg, \cup, \cap$ of $K$ are defined as in $L$ logic, i.e. for any $a, b \in \{F, \bot, T\},$

$$\neg \bot = \bot, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = \max\{a, b\},$$

$$a \cap b = \min\{a, b\}.$$

**Implication** in Kleene’s logic is defined as follows.

For any $a, b \in \{F, \bot, T\},$

$$a \Rightarrow b = \neg a \cup b.$$

3. Write a formal proof in $S$ with 2 applications of the rule $(r).$

**QUESTION 4** Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))).$$
QUESTION 5

$H$ is the following proof system:

$H = ( \mathcal{L}_{\rightarrow, \neg}, \mathcal{F}, AX = \{A1, A2, A3, A4\}, MP )$

A1  $(A \Rightarrow (B \Rightarrow A))$,
A2  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
A3  $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,
A4  $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP  (Rule of inference)

\[
\frac{A \quad (A \Rightarrow B)}{B} (MP)
\]

(1) Prove that $H$ is SOUND under classical semantics.

(2) Does Deduction Theorem holds for $H$? Justify shortly your answer.

(3) Is $H$ COMPLETE with respect to all classical semantics tautologies? JUSTIFY your answer.

QUESTION 6  Let $H$ be the proof system defined in QUESTION 1.

(a) Prove the following: $A \vdash_H (A \Rightarrow A)$
(b) We know that $\vdash_{H} (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash H B$.

**QUESTION 7** Here are consecutive steps $B_1, ..., B_5$ in the formal proof in $H_2$ of $(B \Rightarrow \neg \neg B)$

$B_1 = ((\neg \neg \neg B \Rightarrow \neg B) \Rightarrow ((\neg \neg \neg B \Rightarrow B) \Rightarrow \neg \neg B))$

$B_2 = (\neg \neg \neg B \Rightarrow \neg B)$

$B_3 = ((\neg \neg \neg B \Rightarrow B) \Rightarrow \neg \neg B)$

$B_4 = (B \Rightarrow (\neg \neg \neg B \Rightarrow B))$
\[ B_5 = (B \Rightarrow \neg\neg B) \]

**Complete** the steps 

\[ B_1, \ldots, B_5 \]

of the proof by writing all details in the space provided below each step of the proof. 

You have to write down the **proper substitutions and formulas** used at each step of the proof.

**You can use** the following already proved facts:

1. 

\[ (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C), \]

2. 

\[ \vdash_{H_2} (\neg\neg B \Rightarrow B). \]