Chapter 9
Completeness Theorem: Proof 1

We consider a sound proof system (under classical semantics)

\[ S = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \ \mathcal{A}, \ \mathcal{L}, \ \text{MP} ),\]

such that the formulas listed below are provable in \( S \).

1. \((A \Rightarrow (B \Rightarrow A))\),

2. \(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))))\),

3. \(((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))\),
4. \((A \Rightarrow A)\),

5. \((B \Rightarrow \neg\neg B)\),

6. \((\neg A \Rightarrow (A \Rightarrow B))\),

7. \((A \Rightarrow (\neg B \Rightarrow \neg (A \Rightarrow B)))\),

8. \(((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))\),

9. \(((\neg A \Rightarrow A) \Rightarrow A)\),

We present here two proofs of the following theorem.
Completeness Theorem For any formula $A$ of $S$,

$$\models A \text{ if and only if } \vdash_S A.$$  

Observation 1 All the above formulas have proofs in the system $H_2$ and the system $H_2$ is sound, hence the Completeness Theorem for the system $S$ implies the completeness of the system $H_2$.

Observation 2 We have assumed that the system $S$ is sound, i.e. that the following theorem holds for $S$.

Soundness Theorem

For any formula $A$ of $S$,

if $\vdash_S A$, then $\models A$.  

It means that in order to prove the Completeness Theorem we need to prove only the following implication.

For any formula $A$ of $S$,

If $\models A$, then $\vdash_S A$.

Both proofs of the Completeness Theorem rely strongly of the Deduction Theorem, as discussed and proved in the previous chapter.
Deduction theorem was proved for the system $H_1$ that is different that $S$, but all formulas that were used in its proof are provable in $S$, so it is valid for $S$ as well, as it was for the system $H_2$, i.e. the following theorem holds.

**Deduction Theorem for $S$**

For any formulas $A$, $B$ of $S$ and $\Gamma$ be any subset of formulas of $S$,

$$\Gamma, \ A \vdash_S \ B \quad \text{if and only if} \quad \Gamma \vdash_S (A \Rightarrow B).$$
It is possible to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proof in later chapters.

The first proof presented here is similar in its structure to the proof of the deduction theorem and is due to Kalmar, 1935.

It shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof. It is hence called a proof-construction method.

The second proof is a proof of the equivalent opposite implication to the Completeness part, i.e. we show how one can deduce that a formula $A$ is not a tautology from the fact that it does not have a proof. It is hence called a counter-model construction method.
Completeness Theorem

A Proof - Construction Method

We first present one definition and to prove one lemma.

We write $\vdash A$ instead of $\vdash_S A$, as the system $S$ is fixed.

Definition Let $A$ be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$.

Let $v$ be variable assignment $v : VAR \rightarrow \{T, F\}$. 
DEFINITION 1

We define, for \( A, b_1, b_2, \ldots, b_n \) and \( v \) a corresponding formulas \( A', B_1, B_2, \ldots, B_n \) as follows:

\[
A' = \begin{cases} 
A & \text{if } v^*(A) = T \\
\neg A & \text{if } v^*(A) = F 
\end{cases}
\]

\[
B_i = \begin{cases} 
b_i & \text{if } v(b_i) = T \\
\neg b_i & \text{if } v(b_i) = F 
\end{cases}
\]

for \( i = 1, 2, \ldots, n \).
Example 1: let $A$ be a formula $(a \Rightarrow \neg b)$.

Let $v$ be such that

$$v(a) = T, \quad v(b) = F.$$ 

In this case: $b_1 = a$, $b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$.

The corresponding $A', B_1, B_2$ are:

$A' = A$ (as $v^*(A) = T$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).
Example 2

Let $A$ be a formula

$\left( (\neg a \Rightarrow \neg b) \Rightarrow c \right)$

and let $v$ be such that

$v(a) = T, \ v(b) = F, v(c) = F.$

Evaluate $A', B_1, ... B_n$ as defined by the definition 1.
In this case \( n = 3 \) and
\[
b_1 = a, \ b_2 = b, b_3 = c,
\]
and we evaluate
\[
v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) =
\]
\[
(\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c) =
\]
\[
(\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F.
\]
The corresponding $A', B_1, B_2, B_2$ are:

$$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

as $v^*(A) = F$,

$$B_1 = a \quad (\text{as } v(a) = T),$$

$$B_2 = \neg b \quad (\text{as } v(b) = F).$$

$$B_3 = \neg c \quad (\text{as } v(c) = F).$$
The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula $A$ and a variable assignment $v$ a corresponding deducibility relation.

**MAIN LEMMA** For any formula $A$ and a variable assignment $v$, if $A', B_1, B_2, ..., B_n$ are corresponding formulas defined by our definition, then

$$B_1, B_2, ..., B_n \vdash A'.$$
Example 3 Let $A, v$ be as defined by the example 1, then the Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b).$$

Example 4 Let $A, v$ be as defined in example 2, then the lemma asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$
Proof of the MAIN LEMMA  The proof is by induction on the degree of $A$ i.e. a number $n$ of logical connectives in $A$.

Case: $n = 0$

In the case that $n = 0$ $A$ is atomic and so consists of a single propositional variable, say $a$.

Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$.

We obtain that

$$a \vdash a$$

by the Deduction Theorem and the fact that $\vdash (A \Rightarrow A)$, i.e. also $\vdash (a \Rightarrow a)$. 
In case when \( v^*(A) = F \) we have that
\[
A' = \neg A = \neg a, \\
B_1 = \neg a.
\]

We obtain that
\[
\neg a \vdash \neg a
\]
also by the Deduction Theorem and assumption \( \vdash (A \Rightarrow A) \) in \( S \).

This proves that Lemma holds for \( n = 0 \).
Now assume that the lemma holds for any $A$ with $j < n$ connectives.

Prove: lemma holds for $A$ with $n$ connectives.

There are several subcases to deal with.
Case: $A$ is $\neg A_1$

If $A$ is of the form $\neg A_1$ then $A_1$ has less then $n$ connectives.

By the inductive assumption we have the formulas

$$A_1', B_1, B_2, ..., B_n$$

corresponding to the $A_1$ and the propositional variables $b_1, b_2, ..., b_n$ in $A_1$, such that

$$B_1, B_2, ..., B_n \vdash A_1'$$

Observe, that the formulas $A$ and $\neg A_1$ have the same propositional variables.

So the corresponding formulas $B_1, B_2, ..., B_n$ are the same for both of them.
We are going to show that the inductive assumption allows us to prove that the lemma holds for \( A \), ie. that

\[ B_1, B_2, ..., B_n \vdash A'. \]

There two cases to consider.

Case: \( v^*(A_1) = T \)

If \( v^*(A_1) = T \) then by definition

\[ A'_1 = A_1 \]

and by the inductive assumption

\[ B_1, B_2, ..., B_n \vdash A_1 \]
In this case: \( v^*(A) = v^*(-A_1) = \neg v^*(T) = F \)

So we have that \( A' = \neg A = \neg \neg A_1 \).

Since we have assumed about \( S \) that

\[
\vdash (A_1 \Rightarrow \neg \neg A_1)
\]

we obtain by the monotonicity that also

\[
B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow \neg \neg A_1).
\]
By inductive assumption and Modus Ponens we have that also

\[ B_1, B_2, \ldots, B_n \vdash \neg \neg A_1, \]

and as \( A' = \neg A = \neg \neg A_1 \) we get

\[ B_1, B_2, \ldots, B_n \vdash \neg A, \]

\[ B_1, B_2, \ldots, B_n \vdash A'. \]

**Case:** \( v^*(A_1) = F \)

If \( v^*(A_1) = F \) then \( A'_1 = \neg A_1 \) and \( v^*(A) = T \)

so \( A' = A. \)

**Therefore** by the inductive assumption we have that

\[ B_1, B_2, \ldots, B_n \vdash \neg A_1 \]

that is (as \( A = \neg A_1 \))

\[ B_1, B_2, \ldots, B_n \vdash A'. \]
Case: \( A \) is \( (A_1 \Rightarrow A_2) \)

If \( A \) is \( (A_1 \Rightarrow A_2) \) then \( A_1 \) and \( A_2 \) have less than \( n \) connectives.

By the inductive assumption we have
\[
B_1, B_2, ..., B_n \vdash A_1'
\]

and
\[
B_1, B_2, ..., B_n \vdash A_2',
\]

where \( B_1, B_2, ..., B_n \) are formulas corresponding to the propositional variables in \( A \).

Now we have the following subcases to consider.
Case: \( v^*(A_1) = v^*(A_2) = T \)

If \( v^*(A_1) = T \) then \( A_1' \) is \( A_1 \) and if \( v^*(A_2) = T \) then \( A_2' \) is \( A_2 \).
We also have \( v^*(A_1 \Rightarrow A_2) = T \) and so \( A' \) is \( (A_1 \Rightarrow A_2) \).

By the above and the inductive assumption, therefore, \( B_1, B_2, ..., B_n \vdash A_2 \) and since we have assumed about \( S \) that \( \vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2)) \), we have by monotonicity and Modus Ponens, that \( B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2) \), that is \( B_1, B_2, ..., B_n \vdash A' \).
Case: $v^*(A_1) = T, v^*(A_2) = F$

If $v^*(A_1) = T$ then $A_1' = A_1$ and

if $v^*(A_2) = F$ then $A_2' = \neg A_2$.

Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so $A' = \neg (A_1 \Rightarrow A_2)$.

By the above and the inductive assumption, therefore, $B_1, B_2, ..., B_n \vdash \neg A_2$. Since we have assumed $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg (A_1 \Rightarrow A_2)))$, we have by monotonicity and Modus Ponens twice, that $B_1, B_2, ..., B_n \vdash \neg (A_1 \Rightarrow A_2)$, that is

$B_1, B_2, ..., B_n \vdash A'$. 

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Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1' = \neg A_1$ and, whatever value $v$ gives $A_2$, we have that $v^*(A_1 \Rightarrow A_2) = T$ and so $A' = (A_1 \Rightarrow A_2)$.

Therefore,

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

and since $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$, by monotonicity and Modus Ponens we get that

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2),$$

that is

$$B_1, B_2, ..., B_n \vdash A'.$$
With that we have covered all cases and, by induction on $n$, the proof of the lemma is complete.

Proof of the Completeness Theorem

Assume that $\models A$.

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in $A$, i.e. $A = A(b_1, b_2, ..., b_n)$.

By the lemma we know that, for any variable assignment $v$, the corresponding formulas $A', B_1, B_2, ..., B_n$ can be found such that

$$B_1, B_2, ..., B_n \vdash A'$$
Note here that $A'$ of the definition is $A$ for any $v$ since $\models A$.

Hence, if $v$ is such that $v(b_n) = T$, then $B_n$ is $b_n$ and

$$B_1, B_2, ..., b_n \vdash A.$$ 

If $w$ is such that $w(b_n) = F$, then $B_n$ is $\neg b_n$ and by the lemma

$$B_1, B_2, ..., \neg b_n \vdash A.$$ 

So, by the Deduction Theorem, we have

$$B_1, B_2, ..., B_{n-1} \vdash (b_n \Rightarrow A)$$ 

and

$$B_1, B_2, ..., B_{n-1} \vdash (\neg b_n \Rightarrow A).$$
By monotonicity and assumed formula 9

$$\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

we have that

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

Applying Modus Ponens twice we get that

$$B_1, B_2, ..., B_{n-1} \vdash A.$$

Similarly, $$v^*(B_{n-1})$$ may be T or F, and, again applying Deduction Theorem, monotonicity, and $$\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$, and Modus Ponens twice we can eliminate $$B_{n-1}$$ just as we eliminated $$B_n$$.

After n steps, we finally obtain proof of A in $$S$$, i.e. we have that

$$\vdash A.$$