CHAPTER 9

Classical Predicate Logic: Completeness and Deduction Theorems

There are several quite distinct approaches to the Completeness Theorem, corresponding to the ways of thinking about proofs. Within each of the approaches there are endless variations in exact formulation, corresponding to the choice of methods we want to use to proof the Completeness Theorem. Different basic approaches are important, though, for they lead to different applications. We have presented two of the approaches for the propositional logic: Hilbert style formalizations (proof systems) in chapter **??**, and Gentzen style formalizations (automated proof systems) in chapter **??**. We have also presented for each of the approaches methods of proving the completeness theorem. Two proofs of completeness theorem for Hilbert style proof system in chapter **??** and a constructive proofs for several Gentzen style proof systems in chapter **??**.

There are many proofs of the Completeness Theorem for predicate (first order) logic. We present here in a great detail, a version of Henkin's proof as included in a classic *Handbook of Mathematical Logic* (1977). It contains a method for reducing certain problems of first-order logic back to problems about propositional logic. We give independent proof of Compactness Theorem 1 for propositional logic. Reduction to Propositional Logic Theorem 2, Compactness Theorem 3 for first-order logic, Löwenheim-Skolem Theorem 4 and Gödel Completeness Theorem 7 fall out of the Henkin method.

We choose this particular proof of completeness of first order logic not only for it being one of the oldest and most classical, but also for its connection with the propositional logic. Moreover, the proof of the Compactness Theorem 1 is based on semantical version of syntactical notions and techniques crucial to the second proof of completeness theorem for propositional logic covered in chapter **??** and hence is familiar to the reader.

1 Reduction Predicate Logic to Propositional Logic

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a first order language with equality (definition 12). We assume that the sets **P, F, C** are infinitely enumerable. We also assume that it has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}).$$

Our goal now is to define a **propositional logic** within $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$. We do it in a sequence of steps.

First we define a special subset $P\mathcal{F}$ of formulas of $\mathcal{L}$, called a set of all **propositional formulas** of $\mathcal{L}$.

Intuitively these are formulas of $\mathcal{L}$ which are not direct propositional combination of simpler formulas, that are atomic formulas or formulas beginning with quantifiers. Formally, we have the following.

Definition 1 (Prime Formulas)

Any formula from that set $\mathcal{P}$ defined by (1) is called a **prime formula** *of $\mathcal{L}$.*

$$\mathcal{P} = A\mathcal{F} \cup \{\forall xB, \quad B \in \mathcal{F}\} \cup \{\forall xB : \quad B \in \mathcal{F}\}, \tag{1}$$

where the set $A\mathcal{F}$ is the set of all atomic formulas of $\mathcal{L}$.

The set $\mathcal{P} \subseteq \mathcal{F}$, called a set of all prime formulas of $\mathcal{L}$ plays in the propositional logic we define the role

Example 1

The following are primitive formulas.

$R(t_1, t_2), \quad \forall x(A(x) \Rightarrow \neg A(x)), \quad (c = c), \quad \exists x(Q(x, y) \cap \forall yA(y)).$

The following are not primitive formulas.

$(R(t_1, t_2) \Rightarrow (c = c)), \quad (R(t_1, t_2) \cup \forall x(A(x) \Rightarrow \neg A(x)).$

Given a set $\mathcal{P}$ of primitive formulas we define in a standard way the set $P\mathcal{F}$ of *propositional formulas* of $\mathcal{L}$. as follows.

Definition 2 (Propositional Formulas of $\mathcal{L}$)

Let $\mathcal{F}, \mathcal{P}$ be sets of all formulas and prime formulas (1) of $\mathcal{L}$, respectively. The smallest set $P\mathcal{F} \subseteq \mathcal{F}$ such that

(i) $\mathcal{P} \subseteq P\mathcal{F}$,

(ii)If $A, B \in P\mathcal{F}$, then $(A \Rightarrow B), (A \cup B), (A \cap B)$, and $\neg A \in P\mathcal{F}$

is called a set of all **propositional formulas** *of the predicate language $\mathcal{L}$.*

The set $\mathcal{P}$ is called **atomic propositional** *formulas of $\mathcal{L}$.*

Propositional Semantics for $\mathcal{L}$

We define propositional semantics for propositional formulas in $P\mathcal{F}$ as follows.

Definition 3 (Truth assignment)

Let $\mathcal{P}$ be a set of **atomic propositional** *formulas of $\mathcal{L}$ and $\{T, F\}$ be the set of logical values "true" and "false". Any function*

$$v : \mathcal{P} \longrightarrow \{T, F\}$$

is called a **truth assignment** *in $\mathcal{L}$.*

We extend v to the set $P\mathcal{F}$ of **all propositional** formulas of $\mathcal{L}$ by defining the mapping

$$v^* : \ P\mathcal{F} \longrightarrow \{T, F\}$$

as follows.

$v^*(A) = v(A)$ for $A \in \mathcal{P}$,

and for any $A, B \in P\mathcal{F}$,

$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B)$,

$v^*(A \cup B) = v^*(A) \cup v^*(B)$,

$v^*(A \cap B) = v^*(A) \cap v^*(B)$,

$v^*(\neg A) = \neg v^*(A)$.

Definition 4 (Propositional Model)

A truth assignment $v : \mathcal{P} \longrightarrow \{T, F\}$ is called a propositional model *for a formula $A \in P\mathcal{F}$ if and only if $v^*(A) = T$.*

Definition 5 (Propositional Tautology)

For any formula $A \in P\mathcal{F}$,
$A \in P\mathcal{F}$ is a propositional tautology *of $\mathcal{L}$ if and only if*
$v^*(A) = T$ for all $v : \mathcal{P} \longrightarrow \{T, F\}$.

For the sake of simplicity we will often say *model, tautology* instead *propositional model, propositional tautology* for $\mathcal{L}$.

Definition 6 (Model for the Set)

Given a set S of propositional formulas. We say that v is a model for the set S *if and only if v is a model for all formulas $A \in S$.*

Definition 7 (Consistent Set)

A set $S \subseteq P\mathcal{F}$ of propositional formulas of $\mathcal{L}$ is **consistent** *(in a sense of propositional logic) if it has a (propositional) model.*

Definition 8 (Inconsistent Set)

A set $S \subseteq P\mathcal{F}$ of propositional formulas of $\mathcal{L}$ is **inconsistent** *if it has no model.*

Theorem 1 (Compactness Theorem for Propositional Logic of $\mathcal{L}$)

A set $S \subseteq P\mathcal{F}$ of propositional formulas of $\mathcal{L}$ is **consistent** *if and only if every* **finite** *subset of S is* **consistent**.

Proof

Assume that S is a consistent set. By definition 7, it has a model. Tts model is also a model for all its subsets, including all finite subsets, and so all its finite subsets are consistent.

To prove the converse implication, i.e. the **nontrivial half** of the Compactness Theorem we write it in a slightly modified form. To do so, we introduce the following definition.

Definition 9 (Finitely Consistent Set (FC))

Any set S such that all its finite subsets are consistent is called finitely consistent.

We use this definition 9 to re-write the Compactness Theorem as follows.

A set S of propositional formulas of $\mathcal{L}$ is consistent if and only if it is finitely consistent.

The nontrivial half of it still to be proved is now stated now as

Every finitely consistent set of propositional formulas of $\mathcal{L}$ is consistent.

The **proof** of the nontrivial half of the Compactness Theorem 1, as stated above, consists of the following four steps.

S1 We introduce the notion of a *maximal finitely consistent set*.

S2 We show that every *maximal finitely consistent set* is consistent by constructing its model.

S3 We show that every *finitely consistent set S* can be extended to a *maximal finitely consistent set S^**. I.e we show that for every finitely consistent set S there is a set S^*, such that $S \subseteq S^*$ and S^* is maximal finitely consistent.

S4 We use **S2** and **S3** to justify the following reasoning.

Given a *finitely consistent* set S. We extend it, via construction to be defined in the step **S3** to a *maximal finitely consistent* set S^*. By the **S2**, S^* is consistent and hence so is the set S.
This **ends the proof** of the Compactness Theorem 1.

Here are the details and proofs needed for completion of steps **S1 - S4**.

Step **S1**
We introduce the following definition.

Definition 10 (Maximal Finitely Consistent Set (MFC))

Any set $S \subseteq P\mathcal{F}$ is maximal finitely consistent *if it is finitely consistent and for every formula A, either $A \in S$ or $\neg A \in S$.*

We use notation **MFC** for maximal finitely consistent set, and **FC** for the finitely consistent set.

Step **S2**
We prove the following **MFC** lemma **??** and the Property 1.

Lemma 1

Any MFC set is consistent.

Proof
Given a MFC set denoted by S^*. We prove its consistency by constructing model for it, i.e. by constructing a truth assignment $v : \mathcal{P} \longrightarrow \{T, F\}$, such that for all $A \in S^*$, $v^*(A) = T$.

Observe that directly from the definition 10 we have the following property of the the **MFC** sets.

Property 1 (MFC)

For any MFC set S^* and for every $A \in P\mathcal{F}$, exactly one of the formulas A, $\neg A$ belongs to S^*.

In particular, for any $P \in P\mathcal{F}$, we have that exactly one of formulas P, $\neg P$ belongs to S^*. This justifies the correctness of the following definition.

Definition 11

For any **MFC** *set S^*, mapping $v : \mathcal{P} \longrightarrow \{T, F\}$, such that*

$$v(P) = \begin{cases} T & \text{if } P \in S^* \\ F & \text{if } P \notin S^* \end{cases}$$

is called a truth assignment defined by S^*.
We extend v to $v^* : P\mathcal{F} \longrightarrow \{T, F\}$ in a usual way.

We prove now that the truth assignment v defined by S^* (definition 11) is a **model** for S^*, we show for any $A \in P\mathcal{F}$,

$$v^*(A) = \begin{cases} T & \text{if } A \in S^* \\ F & \text{if } A \notin S^* \end{cases}$$

We prove it by induction on the degree of the formula A as follows.

The base case of $A \in \mathcal{P}$ follows immediately from the definition of v.

Case $A = \neg C$ Assume that $A \in S^*$. This means $\neg C \in S^*$ and by **MCF Property** we have that $C \notin S^*$. So by the inductive assumption $v^*(C) = F$ and $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$.

Assume now that $A \notin S^*$. By **MCF** Property **??** we have that $C \in S^*$. By the inductive assumption $v^*(C) = T$ and $v^*(A) = v^*(\neg C) = \neg v^*(T) = \neg T = F$.

This proves that for any formula A,

$$v^*(\neg A) = \begin{cases} T & \text{if } \neg A \in S^* \\ F & \text{if } \neg A \notin S^* \end{cases}$$

Case $A = (B \cup C)$ Let $(B \cup C) \in S^*$. It is enough to prove that in this case $B \in S^*$ and $C \in S^*$, because then from the inductive assumption $v^*(C) = v^*(D) = T$ and $v^*(B \cup C) = v^*(B) \cup v^*(C) = T \cup T = T$.

Assume that $(B \cup C) \in S^*$, $B \notin S^*$ and $C \notin S^*$. Then by**MCF** Property **??** we have that $\neg B \in S^*$, $\neg C \in S^*$ and consequently the set

$$\{(B \cup C), \neg B, \neg C\}$$

is a finite inconsistent subset of S^*, what contradicts the fact that S^* is finitely consistent.

Assume now that $(B \cup C) \notin S^*$. By **MCF** Property **??**, $\neg(B \cup C) \in S^*$ and by the $A = \neg C$ we have that $v^*(\neg(B \cup C)) = T$. But $v^*(\neg(B \cup C)) = \neg v^*((B \cup C)) = T$ means that $v^*((B \cup C)) = F$, what end the proof of this case.

The remaining cases of $A = (B \cap C), A = (B \Rightarrow C)$ are similar to the above and are left to the reader as an exercise.

This **end the proof** of lemma 1 and completes the step **S2**.

S3: Maximal finitely consistent extension

Given a finitely consistent set S, we construct its *maximal finitely consistent extension* S^* as follows.

The set of all formulas of $\mathcal{L}$ is countable, so is $\mathcal{PF}$. We assume that all propositional formulas form a one-to-one sequence

$$A_1, A_2, ..., A_n, ... \tag{2}$$

We define a chain

$$S_0 \subseteq S_1 \subseteq S_2 .... \subseteq S_n \subseteq .... \tag{3}$$

of *extentions* of the set S by

$$S_0 = S;$$

$$S_{n+1} = \begin{cases} S_n \cup \{A_n\} & \text{if } S_n \cup \{A_n\} \text{ is finitely consistent} \\ S_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

We take

$$S^* = \bigcup_{n \in N} S_n. \tag{4}$$

Clearly, $S \subseteq S^*$ and for every A, either $A \in S^*$ or $\neg A \in S^*$. To finish the proof that S^* is MCF we have to show that it is finitely consistent.

First, let observe that if all sets S_n are finitely consistent, so is $S^* = \bigcup_{n \in N} S_n$. Namely, let $S_F = \{B_1, ..., B_k\}$ be a finite subset of S^*. This means that there are sets $S_{i_1}, ...S_{i_k}$ in the chain (3) such that $B_m \in S_{i_m}$, $m = 1, ..k$. Let $M = max(i_1, ...i_k)$. Obviously $S_F \subseteq S_M$ and S_M is finitely consistent as an element of the chain (3). This proves the if all sets S_n are finitely consistent, so is S^*.

Now we have to prove only that all S_n in the chain (3) are finitely consistent. We carry the proof by induction over the length of the chain. $S_0 = S$, so it is FC by assumption of the Compactness Theorem 1. Assume now that S_n is FC, we prove that so is S_{n+1}. We have two cases to consider.

Case 1 $S_{n+1} = S_n \cup \{A_n\}$, then S_{n+1} is FC by the definition of the chain (3).

Case 2 $S_{n+1} = S_n \cup \{\neg A_n\}$. Observe that this can happen only if $S_n \cup \{A_n\}$ is not FC, i.e. there is a finite subset $S_n^{'} \subseteq S_n$, such that $S_n^{'} \cup \{A_n\}$ is not consistent.

Suppose now that S_{n+1} is not FC. This means that there is a finite subset $S_n^{''} \subseteq S_n$, such that $S_n^{''} \cup \{\neg A_n\}$ is not consistent.

Take $S_n^{'} \cup S_n^{''}$. It is a finite subset of S_n so is consistent by the inductive assumption. Let v be a model of $S_n^{'} \cup S_n^{''}$. Then *one* of $v^*(A), v^*(\neg A)$ *must be* T. This contradicts the inconsistency of both $S_n^{'} \cup \{A_n\}$ and $S_n^{'} \cup \{\neg A_n\}$.

Thus, in ether case, S_{n+1}, is after all consistent.

This **completes** the proof of the step **S3** and the proof of the **compactness theorem** for propositional logic of $\mathcal{L}$ (theorem 1) via the argument presented in the step **S4**.

1.1 Henkin Method for Reduction of Predicate Logic to Propositional Logic

Propositional tautologies within $\mathcal{L}$, as defined here (definition 5) barely scratch the surface of the collection of predicate (first -order) tautologies, i.e. of the predicate *valid* formulas, as they are often called. For example the following first-order formulas are propositional tautologies,

$$(\exists x A(x) \cup \neg \exists x A(x)),$$
$$(\forall x A(x) \cup \neg \forall x A(x)),$$
$$(\neg(\exists x A(x) \cup \forall x A(x)) \Rightarrow (\neg \exists x A(x) \cap \neg \forall x A(x))),$$

but the following are predicate (first order) tautologies (valid formulas) that are not propositional tautologies:

$$\forall x (A(x) \cup \neg A(x)),$$
$$(\neg \forall x A(x) \Rightarrow \exists x \neg A(x)).$$

The first formula above is just a prime formula, the second is of the form $(\neg B \Rightarrow C)$, for B and C prime.

To stress the difference between the propositional and predicate (first order) tautologies some books reserve the word *tautology* for the propositional tautologies alone, using the notion of *valid formula* for the predicate (first order) tautologies. We use here both notions, with the preference to *predicate tautology* or *tautology* for short when there is no room for misunderstanding.

To make sure that there is no misunderstandings we remind the following definitions from chapter **??**.

Given a first order language $\mathcal{L}$ with the set of variables VAR and the set of formulas $\mathcal{F}$. Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$, with the universe M and the interpretation I and let $s : VAR \longrightarrow M$ be an assignment of $\mathcal{L}$ in M. We bring back some basic definitions from Chapter **??**

A is satisfied in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **satisfied** in $\mathcal{M}$ if there is an assignment $s : VAR \longrightarrow M$ such that

$$(\mathcal{M}, s) \models A.$$

A is true in $\mathcal{M}$

Given a structure $\mathcal{M} = [M, I]$, we say that a formula A is **true** in $\mathcal{M}$ if $(\mathcal{M}, s) \models A$ for all assignments $s : VAR \longrightarrow M$.

Model $\mathcal{M}$

If A is **true** in a structure $\mathcal{M} = [M, I]$, then $\mathcal{M}$ is called a **model** for A. We denote it as

$$\mathcal{M} \models A.$$

A is predicate tautology (valid)

A formula A is a predicate tautology (valid) if it is true in all structures $\mathcal{M} = [M, I]$, i.e. if all structures are models of A.

We use use the term **predicate tautology** and and denote it, when there is no confusion with propositional case as

$$\models A.$$

Case: A is a sentence

If A is a sentence, then the truth or falsity of $(\mathcal{M}, s) \models A$ is completely independent of s. Thus we write

$$\mathcal{M} \models A$$

and read $\mathcal{M}$ *is a model of* A, if for some (hence every) valuation s, $(\mathcal{M}, s) \models A$.

Model of a set of sentences

$\mathcal{M}$ is a model of a set S of sentences if and only if $\mathcal{M} \models A$ for all $A \in S$. We write it

$$\mathcal{M} \models S.$$

Predicate and Propositional Models

The relationship between the predicate models that are defined in terms of structures $\mathcal{M} = [M, I]$ and assignments $s : VAR \longrightarrow M$ and propositional models that are defined in terms of truth assignments $v : \mathcal{P} \longrightarrow \{T, F\}$ is established by the following lemma.

Lemma 2

Let $\mathcal{M} = [M, I]$ be a structure for the language $\mathcal{L}$ and let $s : VAR \longrightarrow M$ an assignment in $\mathcal{M}$. There is a truth assignments $v : \mathcal{P} \longrightarrow \{T, F\}$ such that for all formulas A of $\mathcal{L}$,

$$(\mathcal{M}, s) \models A \quad if \ and \ only \ if \quad v^*(A) = T.$$

In particular, for any set S of sentences of $\mathcal{L}$,
if $\mathcal{M} \models S$ then S is consistent in sense of propositional logic.

Proof For any prime formula $A \in P$ we define

$$v(A) = \begin{cases} T & \text{if } (\mathcal{M}, s) \models A \\ F & \text{otherwise.} \end{cases}$$

Since every formula in $\mathcal{L}$ is either prime or is built up from prime formulas by means of propositional connectives, the conclusion is obvious.

Observe, that the converse of the lemma is far from true. Consider a set

$$S = \{\forall x(A(x) \Rightarrow B(x)), \forall x A(x), \exists x \neg B(x)\}.$$

All formulas of S are different prime formulas, S is hence consistent in the sense of propositional logic and obviously has no predicate (first-order) model.

Definition 12 (Language with Equality)

A predicate language

$$\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$$

is called a first order (predicate) language with equality if we one it its predicate symbols is a two argument symbol $E \in \mathbf{P}$ representing an identity relation.

We write $t = s$ as the abbreviation of $E(t, s)$ for any terms $t, s \in \mathbf{T}$ and $t \neq s$ as the abbreviation of $\neg E(t, s)$.

Let $\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a predicate (first order) language with equality. We adopt a following set of axioms.

Equality Axioms (5)

For any free variable or constant of $\mathcal{L}$, i.e for any $u, w, u_i, w_i \in (VAR \cup \mathbf{C})$,

E1 $u = u$,

E2 $(u = w \Rightarrow w = u)$,

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$,

E4 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$,

E5 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$,

where $R \in \mathbf{P}$ and $t \in \mathbf{T}$, i.e. R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary term of $\mathcal{L}$.

Observe that given any structure $\mathcal{M} = [M, I]$. We have by simple verification that for all $s : VAR \longrightarrow M$, and for all $A \in \{E1, E2, E3, E4, E5\}$,

$$(\mathcal{M}, s) \models A.$$

This proves the following

Fact 1

All equality axioms are predicate tautologies (valid) of $\mathcal{L}$.

This is why we still call logic with equality axioms added to it, a logic.

Henkin's Witnessing Expansion of $\mathcal{L}$

Now we are going to define notions that are fundamental to the Henkin's technique for reducing predicate logic to propositional logic. The first one is that of witnessing expansion of the language $\mathcal{L}$.

We construct an expansion of the language $\mathcal{L}$ by adding a set C of new constants to it, i.e. by adding a specially constructed the set C to the set $\mathbf{C}$ such that $C \cap \mathbf{C} = \emptyset$. The construction of the expansion is described below. The language such constructed is called witnessing expansion of the language $\mathcal{L}$.

Definition 13

For any predicate language $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$, the language

$$\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C))$$

for the set C defined by (8) and $\mathcal{L}(C)$ defined by (9) and the construction described below is called a **witnessing expansion** *of $\mathcal{L}$. We write also*

$$\mathcal{L}(C) = \mathcal{L} \cup C.$$

Construction of the witnessing expansion of $\mathcal{L}$

We define the set C of new constants by constructing an infinite sequence

$$C_0, C_1, ..., C_n, ... \tag{6}$$

of sets of constants together with an infinite sequence

$$\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n, ... \tag{7}$$

We define sequences (6), (7) as follows. Let

$$C_0 = \emptyset, \quad \mathcal{L}_0 = \mathcal{L} \cup C_0 = \mathcal{L}.$$

We denote by

$$A[x]$$

the fact that the formula A has exactly one free variable and for each such a formula we introduce a distinct new constant denoted by

$$c_{A[x]}.$$

We define

$$C_1 = \{c_{A[x]} : \quad A[x] \ \in \mathcal{L}_0\}, \quad \mathcal{L}_1 = \mathcal{L} \cup C_1.$$

Assume that we have defined C_n and $\mathcal{L}_n$. We assign distinct new constant symbol $c_{A[x]}$ to each formula $A[x]$ of $\mathcal{L}_n$ which is not already a formula of $\mathcal{L}_{n-1}$ (i.e., if some constant from C_n appears in $A[x]$). We write it informally as $A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})$. We define $C_{n+1} = C_n \cup \{c_{A[x]} : \quad A[x] \in (\mathcal{L}_n - \mathcal{L}_{n-1})\}$ and $\mathcal{L}_{n+1} = \mathcal{L} \cup C_{n+1}$. We put

$$C = \bigcup C_n \tag{8}$$

and

$$\mathcal{L}(C) = \mathcal{L} \cup C. \tag{9}$$

Definition 14 (Henkin Axioms)

The following sentences

H1 $\quad (\exists x A(x) \Rightarrow A(c_{A[x]}))$*,*

H2 $\quad (A(c_{\neg A[x]}) \Rightarrow \forall x A(x))$

are called **Henkin axioms** *and for any formula A, a constant $c_{A[x]} \in C$ as defined by (8) called a* **witnessing constant***.*

The informal idea behind the Henkin axioms is the following.

The axiom **H1** says:
If $\exists x A(x)$ is true in a structure, choose an element a satisfying $A(x)$ and give it a new name $c_{A[x]}$.

The axiom **H2** says:
If $\forall x A(x)$ is false, choose a counterexample b and call it by a new name $c_{\neg A[x]}$.

Definition 15 (QuantifiersAxioms)

The following sentences

Q1 $\quad (\forall x A(x) \Rightarrow A(t))$*, t is a closed term of $\mathcal{L}(C)$;*

Q2 $\quad (A(t) \Rightarrow \exists x A(x))$*, t is a closed term of $\mathcal{L}(C)$*

are called **quantifiers axioms***.*

Observe that the quantifiers axioms **Q1, Q2** obviously are predicate tautologies.

Definition 16 (Henkin Set)

Any set of sentences of $\mathcal{L}(C)$ which are either Henkin axioms (definition 14) or quantifiers axioms (definition 15) is called the **Henkin set** *and denoted by*

$$S_{Henkin}.$$

The *Henkin* is obviously not true in every $\mathcal{L}(C)$-structure, but we are going to show that every $\mathcal{L}$ -structure can be turned into an $\mathcal{L}(C)$-structure which is a **model** of S_{Henkin}. Before we do so we need to introduce two new notions.

Reduct and Expansion

> Given two languages $\mathcal{L}$ and $\mathcal{L}^{'}$ such that $\mathcal{L} \subseteq \mathcal{L}^{'}$. Let $\mathcal{M}^{'} = [M, I^{'}]$ be a structure for $\mathcal{L}^{'}$. The structure
>
> $$\mathcal{M} = [M, I^{'} \mid \mathcal{L}]$$
>
> is called the *reduct* of $\mathcal{M}^{'}$ to the language $\mathcal{L}$ and $\mathcal{M}^{'}$ is called the *expansion* of $\mathcal{M}$ to the language $\mathcal{L}^{'}$.

Thus the reduct and the expansion $\mathcal{M}^{'}$ and $\mathcal{M}$ are the same except that $\mathcal{M}^{'}$ assigns meanings to the symbols in $\mathcal{L} - \mathcal{L}^{'}$.

Lemma 3

Let $\mathcal{M} = [M, I]$ be any structure for the language $\mathcal{L}$ and let $\mathcal{L}(C)$ be the witnessing expansion of $\mathcal{L}$. There is an expansion $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ such that $\mathcal{M}^{'}$ is a model of the set S_{Henkin}

Proof In order to define the expansion of $\mathcal{M}$ to $\mathcal{M}^{'}$ we have to define the interpretation $I^{'}$ for the symbols of the language $\mathcal{L}(C) = \mathcal{L} \cup C$, such that $I^{'} \mid \mathcal{L} = I$. This means that we have to define $c_{I'}$ for all $c \in C$. By the definition, $c_{I'} \in M$, so this also means that we have to assign the elements of M to all constants $c \in C$ in such a way that the resulting expansion is a model for all sentences from S_{Henkin}.

The quantifier axioms (definition 15) are predicate tautologies so they are going to be true regardless, so we have to worry only about the Henkin axioms (definition 14). Observe now that if the lemma 3 holds for the Henkin axiom **H1**, then it must hold for the axiom **H2**. Namely, let's consider the axiom **H2**:

$$(A(c_{\neg A[x]}) \Rightarrow \forall x A(x)).$$

Assume that $A(c_{\neg A[x]})$ is true in the expansion $\mathcal{M}^{'}$, i.e. that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$ and that $\mathcal{M}^{'} \not\models \forall x A(x)$. This means that $\mathcal{M}^{'} \models \neg\forall x A(x)$ and by the de Morgan

Laws, $\mathcal{M}^{'} \models \exists x \neg A(x)$. But we have assumed that $\mathcal{M}^{'}$ is a model for $H1$. In particular $\mathcal{M}^{'} \models (\exists x \neg A(x) \Rightarrow \neg A(c_{\neg A[x]}))$, and hence $\mathcal{M}^{'} \models \neg A(c_{\neg A[x]})$ and this contradicts the assumption that $\mathcal{M}^{'} \models A(c_{\neg A[x]})$. Thus if $\mathcal{M}^{'}$ is a model for all axioms of the type **H1**, it is also a model for all axioms of the type **H2**.

We define $c_{I'}$ for all $c \in C = \bigcup C_n$ by induction on n. Let $n = 1$ and $c_{A[x]} \in C_1$. By definition, $C_1 = \{c_{A[x]} : A[x] \in \mathcal{L}\}$. In this case we have that $\exists x A(x) \in \mathcal{L}$ and hence the notion $\mathcal{M} \models \exists x A(x)$ is well defined, as $\mathcal{M} = [M, I]$ is the structure for the language $\mathcal{L}$.

As we consider arbitrary structure $\mathcal{M}$, there are two possibilities: $\mathcal{M} \models \exists x A(x)$ or $\mathcal{M} \not\models \exists x A(x)$.

We define $c_{I'}$, for all $c \in C_1$ as follows.

If $\mathcal{M} \models \exists x A(x)$, then $(\mathcal{M}, v') \models A(x)$ for certain $v'(x) = a \in M$. We set $(c_{A[x]})_{I'} = a$. If $\mathcal{M} \not\models \exists x A(x)$, we set $(c_{A[x]})_{I'}$ arbitrarily.

This makes all the positive Henkin axioms about the $c_{A[x]} \in C_1$ true, i.e. $\mathcal{M} = (M, I) \models (\exists x A(x) \Rightarrow A(c_{A[x]}))$. But once $c_{A[x]} \in C_1$ are all interpreted in M, then the notion $\mathcal{M}^{'} \models A$ is defined for all formulas $A \in \mathcal{L} \cup C_1$. We carry the same argument and define $c_{I'}$, for all $c \in C_2$ and so on. The inductive step in the exactly the same way as the one above.

Definition 17 (Canonical structure)

Given a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$. The expansion $\mathcal{M}^{'} = [M, I^{'}]$ of $\mathcal{M} = [M, I]$ is called a **canonical structure** *for $\mathcal{L}(C)$ if all $a \in M$ are denoted by some $c \in C$. That is,*

$$M = \{c_{I'} : \ c \in C\}.$$

Now we are ready to state and proof a lemma 2 that provides the essential step in the proof of the completeness theorem for predicate logic.

Theorem 2 (The reduction to propositional logic)

Let $\mathcal{L} = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ be a predicate language and let $\mathcal{L}(C) = \mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C} \cup C)$ be a witnessing expansion of $\mathcal{L}$.

For any set S of sentences of $\mathcal{L}$ the following conditions are equivalent.

(i) *S has a model, i.e. there is a structure $\mathcal{M} = [M, I]$ for the language $\mathcal{L}$ such that $\mathcal{M} \models A$ for all $A \in S$.*

(ii) *There is a* **canonical** *$\mathcal{L}(C)$ structure $\mathcal{M} = [M, I]$ which is a model for S, i.e. such that $\mathcal{M} \models A$ for all $A \in S$.*

(iii) *The set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic, where EQ denotes the equality axioms $E1 - E5$.*

Proof The implication $(ii) \to (i)$ is immediate. The implication $(i) \to (iii)$ follows from lemma 3. We have to prove only the implication $(iii) \to (ii)$.

Assume that the set $S \cup S_{Henkin} \cup EQ$ is consistent in sense of propositional logic and let v be a truth assignment to the prime sentences of $\mathcal{L}(C)$, such that $v^*(A) = T$ for all $A \in S \cup S_{Henkin} \cup EQ$. To prove the lemma, we construct a canonical $\mathcal{L}(C)$ structure $\mathcal{M} = [M, I]$ such that, for all sentences A of $\mathcal{L}(C)$,

$$\mathcal{M} \models A \text{ if and only if } v^*(A) = T. \tag{10}$$

The truth assignment v is a propositional model for the set S_{Henkin}, so v^* satisfies the following conditions:

$$v^*(\exists x A(x)) = T \text{ if and only if } v^*(A(c_{A[x]})) = T, \tag{11}$$

$$v^*(\forall x A(x)) = T \text{ if and only if } v^*(A(t)) = T, \tag{12}$$

for all closed terms t of $\mathcal{L}(C)$.

The conditions (11) and (12) allow us to construct the **canonical** $\mathcal{L}(C)$ model $\mathcal{M} = [M, I]$ out of the constants in C in the following way.

To define $\mathcal{M} = [M, I]$ we must (1.) specify the universe M of $\mathcal{M}$, (2.) define, for each n-ary predicate symbol $R \in \mathbf{P}$, the interpretation R_I as an n-argument relation in M, (3.) define, for each n-ary function symbol $f \in \mathbf{F}$, the interpretation $f_I : M^n \to M$, and (4.) define, for each constant symbol c of $\mathcal{L}(C)$, i.e. $c \in \mathbf{C} \cup C$, an element $c_I \in M$.

The construction of $\mathcal{M} = [M, I]$ must be such that the condition (10) holds for for all sentences A of $\mathcal{L}(C)$. This condition (10) tells us how to construct the definitions (1.) - (4.) above. Here are the definitions.

(1.) **Definition** of the universe M of $\mathcal{M}$.

In order to define the universe M we first define a relation $\approx$ on C by

$$c \approx d \text{ if and only if } v(c = d)) = T. \tag{13}$$

The equality axioms axioms guarantee that the relation (13) is equivalence relation on C, i.e. is reflexive, symmetric, and transitive. All axioms are predicate tautologies, so $v(c = d)) = T$ by axiom **E1** and $c \approx c$ holds for any $c \in C$.

Symmetry condition " if $c \approx d$, then $d \approx c$ " holds by axiom **E2**. Assume $c \approx d$, by definition $v(c = d)) = T$. By axiom **E2**

$$v^*((c = d \Rightarrow d = c)) = v(c = d) \Rightarrow v(d = c) = T,$$

i.e. $T \Rightarrow v(d = c) = T$. This is possible only if $v(d = c) = T$. This proves that $d \approx c$.

We prove transitivity in a similar way. Assume now that $c \approx d$ and $d \approx e$. We check to see that $c \approx e$. By the axiom **E3** we have that

$$v^*(((c = d \cap d = e) \Rightarrow c = e)) = T.$$

Since $v(c = d)) = T$ and $v(d = e)) = T$ by $c \approx d$ and $d \approx e$,

$$v^*((c = d \cap d = e) \Rightarrow c = e) = (T \cap T \Rightarrow c = e) = (T \Rightarrow c = e) = T,$$

we get that $v(c = e) = T$ and hence $d \approx e$.
We denote by $[c]$ the equivalence class of c and we define the universe M of $\mathcal{M}$ as

$$M = \{[c] : \ c \in C\}. \tag{14}$$

(2.) **Definition** of $R_I \subseteq M^n$.

Let M be given by (14). We define

$$([c_1], [c_2], \ \ldots, [c_n]) \in R_I \quad \text{if and only if} \quad v(R(c_1, c_2, \ \ldots, c_n)) = T. \tag{15}$$

We have to prove now that R_I is well defined by the condition (15). To do so we must check

$$\text{if} \ \ [c_1] = [d_1], [c_2] = [d_2], \ \ldots, [c_n] = [d_n] \ \text{ and } \ ([c_1], [c_2], \ \ldots, [c_n]) \in R_I,$$

$$\text{then} \quad ([d_1], [d_2], \ \ldots, [d_n]) \in R_I.$$

We have by the axiom **E4** that

$$v^*(((c_1 = d_1 \cap ... \cap c_n = d_n) \Rightarrow (R(c_1, ..., c_n) \Rightarrow R(d_1, ..., d_n)))) = T. \tag{16}$$

By the assumption $[c_1] = [d_1], \ \ldots, [c_n] = [d_n]$ we have that $v(c_1 = d_1) = T, \ldots, v(c_n = d_n) = T$. By the assumption $([c_1], [c_2], \ \ldots, [c_n]) \in R_I$, we have that $v(R(c_1, ..., c_n)) = T$. Hence the condition (16) becomes

$$(T \Rightarrow (T \Rightarrow v(R(d_1, ..., d_n)))) = T.$$

It holds only when $v(R(d_1, ..., d_n)) = T$ and by (15) we proved that

$$([d_1], [d_2], \ \ldots, [d_n]) \in R_I.$$

(3.) **Definition** of $f_I : \ M^n \to M$.

Let $c_1, \ c_2, \ \ldots, \ c_n \in C$ and $f \in \mathbf{F}$. We claim that there is $c \in C$ such that $f(c_1, \ c_2, \ \ldots, \ c_n) = c$ and $v(f(c_1, \ c_2, \ \ldots, \ c_n) = c) = T$.

For consider the formula $A(x)$ given by $f(c_1, \ c_2, \ \ldots, \ c_n) = x$. If $v^*(\exists x A(x)) = v^*(f(c_1, \ c_2, \ \ldots, \ c_n) = x) = T$, we want to prove $v^*(A(c_{A[x]})) = T$, i.e.

$$v(f(c_1, \ c_2, \ \ldots, \ c_n) = c_A) = T.$$

So suppose that $v(f(c_1,\ c_2,\ \dots,\ c_n) = c_A) = F$. But one member of he Henkin set S_{Henkin} (definition 16) is the sentence $(A(f(c_1,\ c_2,\ \dots,\ c_n)) \Rightarrow \exists x A(x))$ so we must have that $v^*(A(f(c_1,\ c_2,\ \dots,\ c_n))) = F$. But this says that v assigns F to the atomic sentence $f(c_1,\ c_2,\ \dots,\ c_n) = f(c_1,\ c_2,\ \dots,\ c_n)$, i.e. By the axiom **E1** $v(c_i = c_i) = T$ for $i = 1, 2 \dots n$ and by **E5**

$$(v^*(c_1 = c_1 \cap \dots\ c_n = c_n) \Rightarrow v^*(f(c_1,\ \dots,\ c_n) = f(c_1, \dots, c_n))) = T.$$

This means that $T \Rightarrow F = T$ and this contradiction proves there is $c \in C$ such that $f(c_1,\ c_2,\ \dots,\ c_n) = c$ and $v(f(c_1,\ c_2,\ \dots,\ c_n) = c) = T$. We can hence define

$$f_I(([c_1], \dots, [c_n]) = [c] \text{ for } c \text{ such that } v(f(c_1, \dots,\ c_n) = c) = T. \tag{17}$$

The argument similar to the one used in (2.) proves that f_I is well defined.

(4.) **Definition** of $c_I \in M$.

For any $c \in C$ we take $c_I = [c]$. If $d \in \mathbf{C}$, then an argument similar to that used on (3.) shows that there is $c \in C$ such that $v(d = c) = T$, i.e. $d \approx c$, so we put $d_I = [c]$.

This completes the construction of the canonical structure $\mathcal{M} = [M, I]$ and guarantees that (10) holds for for all **atomic propositional** sentences (definition 2), i.e. we proved that

$$\mathcal{M} \models B \ \text{ if and only if } \ v^*(B) = T, \text{ for sentences } B \in \mathcal{P}.$$

To complete the proof of the Lemma 2 we prove that the property (10) holds for the canonical structure $\mathcal{M} = [M, I]$ defined above and all other sentences. We carry the proof by induction on length of formulas. The case of propositional connectives is trivial. For example, $\mathcal{M} \models (A \cap B)$ if and only if $\mathcal{M} \models A$ and $\mathcal{M} \models B$) (follows directly from the satisfaction definition) if and only if $v^*(A) = T$ and $v^*(B) = T$ (by the induction hypothesis) if and only if $v^*(A \cap B) = T$. We proved

$$\mathcal{M} \models (A \cap B) \text{ if and only if } v^*(A \cap B) = T,$$

for all sentences A, B of $\mathcal{L}(C)$. The proof for all other connectives is similar.

We prove now the case of a sentence B of the form $\exists x A(x)$, i.e. we want to show that

$$\mathcal{M} \models \exists x A(x) \ \text{ if and only if } \ v^*(\exists x A(x)) = T. \tag{18}$$

$v^*(\exists x A(x)) = T$. Then there is a c such that $v^*(A(c) = T$, so by induction hypothesis, $\mathcal{M} \models A(c)$ so $\mathcal{M} \models \exists x A(x)$.

On the other hand, if $v^*(\exists x A(x)) = F$, then by $S_{Henking}$ quantifier axiom **Q2** (definition 15) we have that $v^*(A(t)) = F$ for all closed terms t of $\mathcal{L}(C)$. In particular, for every $c \in C$ $v^*(A(c)) = F$. By induction hypothesis, $\mathcal{M} \models \neg A(c)$, for all $c \in C$. Since every element of M is denoted by some $c \in C$, $\mathcal{M} \models \neg\exists x A(x)$. Thus we proved (18).

The proof of the case of a sentence B of the form $\forall x A(x)$ is similar and is left to the reader.

The Reduction to Propositional Logic Theorem 2 provides not only a method of constructing models of theories out of symbols, but also gives us immediate proofs of the Compactness Theorem 3 for the predicate logic and Lowenheim-Skolem Theorem 4.

Theorem 3 (Compactness theorem for the predicate logic)

Let S be an y set of predicate formulas of $\mathcal{L}$.
The set S has a model if and only if any finite subset S_0 of S has a model.

Proof
Let S be a set of predicate formulas such that every finite subset S_0 of S has a model. We need to show that S has a model. By the implication $(iii) \rightarrow (i)$ of the Theorem 2 this is equivalent to proving that $S \cup S_{Henkin} \cup EQ$ is consistent in the sense of propositional logic. By the Compactness Theorem 1 for propositional logic of $\mathcal{L}$, it suffices to prove that for every finite subset $S_0 \subset S$, $S_0 \cup S_{Henkin} \cup EQ$ is consistent, which follows from the hypothesis and the implication $(i) \rightarrow (iii)$ of the Reduction to Propositional Logic Theorem 2.

Theorem 4 (Löwenheim-Skolem Theorem)

Let κ be an infinite cardinal and let Γ be a set of at most κ formulas of the first order language.

If the set S has a model, then there is a model $\mathcal{M} = [M, I]$ of S such that $cardM \leq \kappa$.

Proof Let $\mathcal{L}$ be a predicate language with the alphabet $\mathcal{A}$ such that $card(\mathcal{A}) \leq \kappa$. Obviously, $card(\mathcal{F}) \leq \kappa$. By the definition of the witnessing expansion $\mathcal{L}(C)$ of $\mathcal{L}$, $C = \bigcup_n C_n$ and for each n, $card(C_n) \leq \kappa$. So also $cardC \leq \kappa$. Thus any canonical structure for $\mathcal{L}(C)$ has $\leq \kappa$ elements. By the implication $(i) \rightarrow (ii)$ of the Reduction to Propositional Logic Lemma 2 there is a model of S (canonical structure) with $\leq \kappa$ elements.

2 Proof of the Completeness Theorem for Classical Predicate Logic

The proof of Gödel's completeness theorem given by Kurt Gdel in his doctoral dissertation of 1929 (and a rewritten version of the dissertation, published as an article in 1930) is not easy to read today; it uses concepts and formalism that are no longer used and terminology that is often obscure. The version given below attempts to represent all the steps in the proof and all the important ideas faithfully, while restating the proof in the modern language of mathematical logic. This outline should not be considered a rigorous proof of the theorem.

It was first proved by Kurt Gödel in 1929. It was then simplified in 1947, when Leon Henkin observed in his Ph.D. thesis that the hard part of the proof can be presented as the Model Existence Theorem (published in 1949). Henkin's proof was simplified by Gisbert Hasenjaeger in 1953. Other now classical proofs has been published by Rasiowa and Sikorski (1951 1952) using Boolean algebraic methods and by by Beth (1953), using topological methods. Still other proofs may be found in Hintikka(1955) and in Beth(1959).

Hilbert-style Proof System H

Language $\mathcal{L}$

The language $\mathcal{L}$ of the proof system **H** is a predicate (first order) language with equality (definition 12). We assume that the sets **P, F, C** are infinitely enumerable. We also assume that it has a full set of propositional connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}). \tag{19}$$

Logical Axioms LA

The set LA of logical axioms consists of three groups of axioms: propositional axioms PA, equality axioms EA, and quantifiers axioms QA. we write it symbolically as

$$LA = \{PA,\ EA,\ QA\}.$$

For the set PA of *propositional axioms* we choose any complete set of axioms for propositional logic with a full set $\{\neg,\cap,\cap,\Rightarrow\}$ of propositional connectives.

In some formalizations, including the one in the *Handbook of Mathematical Logic, Barwise, ed.* (1977) we base our system **H** on, the authors just say for this group of axioms: "all tautologies". They of course mean all predicate formulas of $\mathcal{L}$ that are substitutions of propositional tautologies. This is done for the need of being able to use freely these predicate substitutions of propositional tautologies in the proof of completeness theorem for the proof system they formalize this way.

In this case these tautologies are listed as axioms of the system and hence are provable in it. This is a convenient approach, but also the one that makes such a proof system not to be finately axiomatizable.

We avoid the infinite axiomatization by choosing a proper finite set of predicate language version of propositional axioms that is known (proved already for propositional case) to be complete, i.e. the one in which all propositional tautologies are provable.

We choose, for name **H** (Hilbert) and historical sake, the set of Hilbert (1928) axioms from chapter **??**.

.For the set EA of *equational axioms* we choose the same set (5) as in section 1.1 because they were used in the proof of Reduction to Propositional Logic Theorem 2 and we want to be able to carry this proof within the system **H**.

For the set QA of *quantifiers axioms* we choose the axioms such that the Henkin set S_{Henkin} axioms **Q1, Q2** are their particular cases, so again a proof of the Reduction to Propositional Logic Theorem 2 can be carried within **H**.

Rules of inference $\mathcal{R}$

There are three inference rules: Modus Ponens (MP) and two quantifiers rules $(G), (G1), (G2)$, called Generalization Rules.

We define the proof system **H** as follows.

$$\mathbf{H} = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R} = \{(MP),\ (G),(G1),\ (G2)\}), \tag{20}$$

where

$\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C})$ is predicate (first order) language with equality (definition 12). We assume that the sets **P, F, C** are infinitely enumerable.

$\mathcal{F}$ is the set of all well formed formulas of $\mathcal{L}$.

LA is the set of logical axioms and

$$LA = \{PA, EA, QA\} \tag{21}$$

for PA, EA, QA defined as follows.

PA is the set of **propositional axioms** (Hilbert, 1928)

A1 $(A \Rightarrow A)$,

A2 $(A \Rightarrow (B \Rightarrow A))$,

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$, A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$,

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A7 $((A \cap B) \Rightarrow A)$,

A8 $((A \cap B) \Rightarrow B)$,

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C)))$,

A10 $(A \Rightarrow (A \cup B))$,

A11 $(B \Rightarrow (A \cup B))$,

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$,

A14 $(\neg A \Rightarrow (A \Rightarrow B))$,

A15 $(A \cup \neg A)$,

for any $A, B, C \in \mathcal{F}$.

EA is the set of **equality axioms**.

E1 $u = u$,

E2 $(u = w \Rightarrow w = u)$,

E3 $((u_1 = u_2 \cap u_2 = u_3) \Rightarrow u_1 = u_3)$,

E4 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (R(u_1, ..., u_n) \Rightarrow R(w_1, ..., w_n)))$,

E5 $((u_1 = w_1 \cap ... \cap u_n = w_n) \Rightarrow (t(u_1, ..., u_n) \Rightarrow t(w_1, ..., w_n)))$,

for any free variable or constant of $\mathcal{L}$, $R \in \mathbf{P}$, and $t \in \mathbf{T}$, where R is an arbitrary n-ary relation symbol of $\mathcal{L}$ and $t \in \mathbf{T}$ is an arbitrary n-ary term of $\mathcal{L}$.

QA is the set of **quantifiers axioms**.

Q1 $(\forall x A(x) \Rightarrow A(t))$,

Q2 $(A(t) \Rightarrow \exists x A(x))$,

where where t is a term, A(t) is a result of substitution of t for all free occurrences of x in $A(x)$, and t is *free for* x in A(x), i.e. no occurrence of a variable in t becomes a bound occurrence in A(t).

$\mathcal{R}$ is the set of **rules of inference**,

$$\mathcal{R} = \{(MP),\ (G),\ (G1),\ (G2)\},$$

where

(MP) is Modus Ponens rule

$$(MP) \quad \frac{A \ ; \ (A \Rightarrow B)}{B},$$

for any formulas $A, B \in \mathcal{F}$.

$(G), (G1), (G2)$ are the following **quantifiers generalization rules**.

$$(G) \quad \frac{A}{\forall x A},$$

where $A \in \mathcal{F}$. In particular we write

$$(G) \quad \frac{A(x)}{\forall x A(x)}$$

for $A(x) \in \mathcal{F}$ and $x \in VAR$.

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))},$$

where $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is not free in B.

$$(G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)},$$

where $A(x), B \in \mathcal{F}$, $x \in VAR$, and B is such that x is not free in B.

We define, as we do for any proof system, a notion of a *proof of a formula A from a set S* of formulas in $\mathbf{H}$ as a finite sequence of formulas $B_1, B_2, \ \ldots \ B_n$, with $B_n = A$, each of which is either a logical axiom of $\mathbf{H}$, a member of S, or else follows from earlier formulas in the sequence by one of the inference rules from $\mathcal{R}$. We write it formally as follows.

Definition 18 (Proof from Γ)

Let $\Gamma \subseteq \mathcal{F}$ be any set of formulas of $\mathcal{L}$. A proof in $\mathbf{H}$ of a formula $A \in \mathcal{F}$ from s set Γ of formulas is a sequence

$$B_1, B_2, \ \ldots \ B_n$$

of formulas, such that

$$B_1 \in LA \cup \Gamma, \qquad B_n = A$$

and for each i, $1 < i \leq n$, either $B_i \in LA \cup \Gamma$ or B_i is a conclusion of some of the preceding expressions in the sequence $B_1, B_2, \ \ldots \ B_n$ by virtue of one of the rules of inference from $\mathcal{R}$.

We write

$$\Gamma \vdash_{\mathbf{H}} A$$

to denote that the formula A *has a proof from* Γ *in* **H** *and*

$$\Gamma \ \vdash \ A,$$

when the proof system **H** *is fixed.*

The case when $\Gamma = \emptyset$ is a special one. By the definition 18, $\emptyset \vdash_{\mathbf{H}} A$ means that in the proof of A only logical axioms LA are used. We hence write

$$\vdash_{\mathbf{H}} A$$

to denote that a formula A has a proof in **H**.

As we work with a fixed (and only one) proof system, we use the notation

$$\Gamma \ \vdash \ A \ \text{ and } \ \vdash \ A$$

to denote the proof of a formula A from a set Γ and proof of a formula A in **H**, respectively.

Any proof of the completeness theorem for a given proof system consists always of two parts. First we have show that *all formulas that have a proof in the system are tautologies.* This implication is called a soundness theorem, or soundness part of the completeness theorem.

The second implication says: *if a formula is a tautology then it has a proof in the proof system.* This alone is sometimes called a completeness theorem (on assumption that the system is sound). Traditionally it is called a completeness part of the completeness theorem.

We know that all axioms of **H** are predicate tautologies (proved in chapter **??** and all rules of inference from $\mathcal{R}$ are sound as the corresponding formulas were also proved in chapter **??** to be predicate tautologies and so the system **H** is sound, i.e. the following holds for **H**.

Theorem 5 (Soundness Theorem)

For every formula $A \in \mathcal{F}$ *of the language* $\mathcal{L}$ *of the proof system* **H**,

$$\textit{if } \vdash A, \quad \textit{then} \quad \models A.$$

The soundness theorem proves that the proofs in the system "produce" only tautologies. We show here, as the next step that our proof system **H** "produces" not only tautologies, but that all tautologies are provable in it.

This is called a *completeness theorem* for classical predicate (first order logic, as it all is proven with respect to classical semantics. This is why it is called a completeness of predicate logic theorem.

The goal is now to prove the completeness part of the following.

Theorem 6 (Gödel Completeness of Predicate Logic)

For any formula A of the language $\mathcal{L}$ of the system **H**,
A is provable in **H** *if and only if A is a predicate tautology (valid).*
We write it symbolically as

$$\vdash A \quad \textit{if and only if} \quad \models A. \tag{22}$$

We are going to prove the Gödel' s Theorem 6 as a particular case of Theorem 7 that follows. It is its more general, and more modern version. This version, as well as the method of proving it, was first introduced by Henkin in 1947. It became with its consequent improvements, as classical as the Gödel's own. It uses the notion of a *logical implication*, and some other notions. We introduce them below.

Definition 19 (Sentence, Closure)

Any formula of $\mathcal{L}$ without free variables is called a **sentence**.
For any formula $A(x_1, \ldots x_n)$, a sentence

$$\forall x_1 \forall x_2 \ldots \forall x_n \ A(x_1, \ldots x_n)$$

is called a **closure** *of $A(x_1, \ldots x_n)$.*

Directly from the definition 19 have that the following hold.

Fact 2 *For any formula $A(x_1, \ldots x_n)$,*

$$\models A(x_1, \ldots x_n) \quad \textit{if and only if} \quad \models \forall x_1 \forall x_2 \ldots \forall x_n \ A(x_1, \ldots x_n).$$

Definition 20 (Logical Implication)

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$, we say that the set Γ **logically implies** *the formula A and write it as $\Gamma \models A$ if and only if all models of Γ are models of A.*

Observe, that in order to prove that $\Gamma \models B$ we have to show that the implication

$$\text{if } \ \mathcal{M} \models \Gamma \ \text{ then } \ \mathcal{M} \models B$$

holds for all structures $\mathcal{M} = [U, I]$ for $\mathcal{L}$.

Lemma 4 *Let Γ be a set of sentences of $\mathcal{L}$, for any formula $A(x_1, \ldots x_n)$ that is not a sentence,*

$$\Gamma \vdash A(x_1, \ldots x_n) \quad \textit{if and only if} \quad \Gamma \models \forall x_1 \forall x_2 \ldots \forall x_n \ A(x_1, \ldots x_n).$$

Proof
Let $B_1, B_2, \ldots B_n$ be a proof of $A(x_1, \ldots x_n)$ from Γ and let $\mathcal{M}$ be a model of Γ. We use Fact 2 and prove by induction on n, that $\mathcal{M} \models \forall x_1 \forall x_2 \ldots \forall x_n\ B_n(x_1, \ldots x_n)$, and hence $\mathcal{M} \models \forall x_1 \forall x_2 \ldots \forall x_n\ A(x_1, \ldots x_n)$. The converse implication is obvious.

Fact 2 and Lemma 4 show that we need to consider only sentences (closed formulas) of $\mathcal{L}$, since a formula of $\mathcal{F}$ is a tautology if and only if its closure is a tautology and is provable from Γ if and only if its closure is provable from Γ. This justifies the following generalization of the original Gödel' s completeness of predicate logic Theorem 6.

Theorem 7 (Gödel Completeness Theorem)

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system **H**.
A sentence A is provable from Γ in **H** *if and only if the set Γ* **logically implies** *A.*
We write it in symbols,

$$\Gamma \vdash A \quad \textit{if and only if} \quad \Gamma \models A. \tag{23}$$

Remark
We want to remind the readers that the Reduction Predicate Logic to Propositional Logic Section 1 is an integral and the first part of the proof the Gödel Completeness Theorem 7. We presented it separately for two reasons.

R1. The reduction method and theorems and their proofs are purely semantical in their nature and hence are independent of the proof system **H**.

R2. Because of **R1.** the reduction method can be used/adapted to a proof of completeness theorem of any other proof system one needs to prove the classical completeness theorem for. See section **??**.

In order to prove it we must formulate it properly so we need to introduce few new important and classical notion and prove some lemmas needed for the proof. The first is the notion of **consistency**.

There are **two definitions of consistency**; semantical and syntactical. The **semantical** one uses definition the notion of a model and says, in plain English: *a set of formulas is consistent if it has a model.*

The **syntactical** one uses the notion of provability and says: *a set of formulas is consistent if one can't prove a contradiction from it.*

We have used, in the Proof Two of the Completeness Theorem for propositional logic (chapter **??**) the **syntactical** definition of consistency. We use here the following **semantical** definition.

Definition 21 (Consistent/Inconsistent)

A set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ is **consistent** *if and only if it has a model, otherwise, it is* **inconsistent***.*

Directly from the above definitions we have the following.

Lemma 5

For any set $\Gamma \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$ and any $A \in \mathcal{F}$,

if $\Gamma \models A$, then the set $\Gamma \cup \{\neg A\}$ is **inconsistent***.*

Proof
Assume $\Gamma \models A$ and $\Gamma \cup \{\neg A\}$ is consistent. By definition 21 there is a structure $\mathcal{M} = [U, I]$, such that $\mathcal{M} \models \Gamma$ and $\mathcal{M} \models \neg A$, i.e. $\mathcal{M} \not\models A$. This is a contradiction with $\Gamma \models A$.

Now we are going to prove the following Lemma 6 that is crucial, together with the Reduction to Propositional Logic Theorem 2 and the above Lemma 5 to the proof of the Completeness Theorem 7.

Lemma 6

Let Γ be any set of sentences of a language $\mathcal{L}$ of Hilbert proof system **H***.*
The following conditions hold For any formulas $A, B \in \mathcal{F}$ of $\mathcal{L}$.

(i) If $\Gamma \vdash (A \Rightarrow B)$ and $\Gamma \vdash (\neg A \Rightarrow B)$, then $\Gamma \vdash B$.

(ii) If $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$, then $\Gamma \vdash (\neg A \Rightarrow B)$ and $\Gamma \vdash (C \Rightarrow B)$.

(iii) If x does not appear in B and if $\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$, then $\Gamma \vdash B$.

(iv) If x does not appear in B and if $\Gamma \vdash ((A(x) \Rightarrow \forall y A(y)) \Rightarrow B)$, then $\Gamma \vdash B$.

Proof
(i) Notice that the formula $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ is a substitution of a propositional tautology, hence by definition of **H**, is provable in it. By monotonicity,

$$\Gamma \vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

By assuption $\Gamma \vdash (A \Rightarrow B)$ and Modus Ponens we get

$$\Gamma \vdash ((\neg A \Rightarrow B) \Rightarrow B).$$

By assuption $S\Gamma \vdash (\neg A \Rightarrow B)$ and Modus Ponens we get $\Gamma \vdash B$.

(ii) The formulas (1) $(((A \Rightarrow B) \Rightarrow (\neg A \Rightarrow B)))$ and (2) $(((A \Rightarrow B) \Rightarrow B) \Rightarrow (C \Rightarrow B))$ are substitution of a propositional tautologies, hence are provable in **H**. Assume $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$. By monotonicity and (1) we get $\Gamma \vdash (\neg A \Rightarrow B)$ and by (2) we get $\vdash (C \Rightarrow B)$.

(iii) Assume $\Gamma \vdash ((\exists y A(y) \Rightarrow A(x)) \Rightarrow B)$. Observe that it is a particular case of assumption $\Gamma \vdash ((A \Rightarrow C) \Rightarrow B)$ in (ii), for $A = \exists y A(y),\ C = A(x), B = B$. Hence by (ii) we have that $\Gamma \vdash (\neg\exists y A(y) \Rightarrow B)$ and $\Gamma \vdash (A(x) \Rightarrow B)$.

Apply Generalization Rule G2 to $\Gamma \vdash (A(x) \Rightarrow B)$ and we have $\Gamma \vdash (\exists y A(y) \Rightarrow B.)$ Then by (i) applied to $\Gamma \vdash (\exists y A(y) \Rightarrow B)$ and $\Gamma \vdash (\neg\exists y A(y) \Rightarrow B)$ we get $\Gamma \vdash B$.

The proof of (iv) is similar to (iii), but uses the Generalization Rule G1.This **ends the proof** of the lemma.

Now we are ready to conduct the proof of the Completeness Theorem for **H**. There are two versions. Theorem 7 that is Gödel original formulation and the one we used in previous chapters of the book. It follows from theorem **??**). We put them both together as follows.

Theorem 8 (H Completeness)

Let Γ be any set of sentences and A any sentence of a language $\mathcal{L}$ of Hilbert proof system **H**.

$$\Gamma \vdash A \quad \textit{if and only if} \quad \Gamma \models A. \tag{24}$$

In particular, for any formula A of $\mathcal{L}$,

$$\vdash A \quad \textit{if and only if} \quad \models A. \tag{25}$$

Proof
We first prove the completeness part (24), i.e. we prove the implication

$$\text{if } \ \Gamma \models A, \ \text{then } \ \Gamma \vdash A. \tag{26}$$

Suppose that $\Gamma \models A$, i.e. we assume that all $\mathcal{L}$ models of Γ are models of A. By Lemma 5 the set $\Gamma \cup \{\neg A\}$ is inconsistent.

Let $\mathcal{M} \models \Gamma$. We construct, as a next step, a witnessing expansion language $\mathcal{L}(C)$ of $\mathcal{L}$ (definition 13). By the Reduction to Propositional Logic Theorem 2, the set $\Gamma \cup S_{Henkin} \cup EQ$ is consistent in a sense of propositional logic in $\mathcal{L}$. The set S_{Henkin} is a Henkin Set (definition 16) and **EQ** are equality axioms (5) that are also the equality axioms EQ of **H**.

By the Compactness Theorem 1 for propositional logic of $\mathcal{L}$ there is a finite set $S_0 \subseteq \Gamma \cup S_{Henkin} \cup EQ$ such that $S_0 \cup \{\neg A\}$ is inconsistent in the sense of propositional logic.

We list all elements of S_0 in a sequence

$$A_1,\ A_2, \ldots, A_n,\ B_1,\ B_2, \ldots, B_m \tag{27}$$

as follows. The sequence $A_1,\ A_2, \ldots, A_n$ consists of those elements of S_0 which are either in $\Gamma \cup EQ$ or else are quantifiers axioms (definition 15) that are particular cases of the quantifiers axioms QA of $\mathbf{H}$. We list them in any order.

The sequence $B_1,\ B_2, \ldots, B_m$ consists of elements of S_0 which are Henkin Axioms (definition 14) but listed carefully as to be described as follows. Observe that by definition 13,

$$\mathcal{L}(C) = \bigcup_{n \in N} \mathcal{L}_n, \text{ for } \mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ \ldots .$$

We define the *rank* of $A \in \mathcal{L}(C)$ to be the least n, such that $A \in \mathcal{L}_n$.

Now we choose for B_1 a Henkin Axiom in S_0 of the *maximum rank*.
We choose for B_1 a Henkin Axiom in $S_0 - \{B_1\}$ of the *maximum rank*.
We choose for B_2 a Henkin Axiom in $S_0 - \{B_1, B_2\}$ of the *maximum rank*, etc.
The point of choosing the formulas B_i's in this way is to make sure that the witnessing constant about which B_i speaks, does not appear in $B_{i+1},\ B_{i+2},\ \ldots, B_m$.
For example, if B_1 is

$$(\exists x C(x) \Rightarrow C(c_{C[x]})),$$

then $C[x]$ does not appear in any of the other $B_2,\ \ldots, B_m$, by the maximality condition on B_1.

We know that that $S_0 \cup \{\neg A\}$ is inconsistent in the sense of propositional logic, i.e. it does not have a (propositional) model. This means that $v^*(\neg A) \neq T$ for all v and so $v^*(A) = T$ for all v. Hence a sentence

$$(A_1 \Rightarrow (A_2 \Rightarrow \ldots (A_n \Rightarrow (B_1 \Rightarrow \ldots (B_m \Rightarrow A))..)$$

is a propositional tautology.
We now replace each witnessing constant in this sentence by a distinct new variable and write the result as

$$({A_1}' \Rightarrow ({A_2}' \Rightarrow \ldots ({A_n}' \Rightarrow ({B_1}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..)$$

. We have $A' = A$ since A has no witnessing constant in it. The result is still a tautology and hence is provable in $\mathbf{H}$ from propositional axioms PA and Modus Ponens. By monotonicity

$$S_0 \vdash ({A_1}' \Rightarrow ({A_2}' \Rightarrow \ldots ({A_n}' \Rightarrow ({B_1}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..). \tag{28}$$

Each of ${A_1}', {A_2}', \ldots, {A_n}'$ is either a quantifiers axiom from QA of $\mathbf{H}$ or else in S_0, so

$$S_0 \vdash {A_i}' \ \text{ for all } \ 1 \leq i \leq n.$$

We apply Modus Ponens to the above and (28) n times and get

$$S_0 \vdash ({B_1}' \Rightarrow ({B_2}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..). \tag{29}$$

For example, if ${B_1}'$ is $(\exists x C(x) \Rightarrow C(x))$, we have by (29)

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow B). \tag{30}$$

for $B = ({B_2}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..)$. By the Reduction to Propositional Logic Theorem 2 part (iii), we get $S_0 \vdash B$, i.e.

$$S_0 \vdash ({B_2}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..). \tag{31}$$

If, for example, ${B_2}'$ is $(D(x) \Rightarrow \forall x D(x))$, we have by (30)

$$S_0 \vdash ((\exists x C(x) \Rightarrow C(x)) \Rightarrow D). \tag{32}$$

for $D = ({B_3}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..)$. By the Reduction to Propositional Logic Theorem 2 part (iv), we get $S_0 \vdash D$, i.e.

$$S_0 \vdash ({B_3}' \Rightarrow \ldots ({B_m}' \Rightarrow A))..). \tag{33}$$

. We hence apply parts (iii) ad (iv) of Theorem 2 to successively remove all ${B_1}', ..., {B_m}'$ and obtain the proof of A from S_0.
This **ends the proof** that $\Gamma \vdash A$ and hence the proof of the completeness part of (24).

The soundness part of of (24), i.e. the implication

$$\text{if } \ \Gamma \vdash A, \ \text{then } \ \Gamma \models A,$$

holds for any sentence A of $\mathcal{L}$ directly by Fact 2, Lemma 4, and Theorem 5.

The Theorem 6, as expressed by (25) follows from Fact 2, Lemma 4 as a case of (24) for $\Gamma = \emptyset$.

This **ends the proof** of Theorem 8 as well as Theorem 7, and the proof of the original Gödel Completeness of Predicate Logic Theorem 6.

2.1 Deduction Theorem

In mathematical arguments, one often assumes a statement A on the assumption (hypothesis) of some other statement B and then concludes that we have proved the implication "if A, then B". This reasoning is justified by the following theorem, called a Deduction Theorem. It was first formulated and proved for a certain Hilbert proof system S for the classical propositional logic by Herbrand in 1930 in a form stated below.

Theorem 9 (Deduction Theorem) *(Herbrand,1930)*

For any formulas A, B of the language of a propositional proof system S,

$$\textit{if} \quad A \vdash_S B, \quad \textit{then} \ \vdash_S (A \Rightarrow B).$$

In chapter **??** we formulated and proved the following, more general version of the Herbrand Theorem 10 for a very simple (two logical axioms and Modus Ponens) propositional proof system $H1$.

Theorem 10 (Deduction Theorem)

For any subset Γ *of the set of formulas* $\mathcal{F}$ *of* H_1 *and for any formulas* $A, B \in \mathcal{F}$*,*

$$\Gamma,\ A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \ \ if\ and\ only\ if\ \ \vdash_{H_1} (A \Rightarrow B).$$

A natural question arises: does deduction theorem holds for the predicate logic in general and for its proof system **H** we defined here?.

The Theorem 10 cannot be carried *directly* to the predicate logic, but it nevertheless holds with *some modifications.* Here is where the problem lays.

Fact 3 *Given the proof system (20), i.e.*
$\boldsymbol{H} = (\mathcal{L}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R} = \{(MP),\ (G), (G1),\ (G2)\})$.
For any formula $A(x) \in \mathcal{F}$*,*
$A(x) \vdash \forall x A(x)$*, but it is not always the case that* $\vdash (A(x) \Rightarrow \forall x A(x))$.

Proof
Obviously, $A(x) \vdash \forall x A(x)$ by Generalization rule (G). Let now $A(x)$ be an atomic formula $P(x)$. By the Completeness Theorem 6, $\vdash (P(x) \Rightarrow \forall x P(x))$ if and only if $\models (P(x) \Rightarrow \forall x P(x))$. Consider a structure $\mathcal{M} = [M, I]$, where M contains at least two elements c and d. We define $P_I \subseteq M$ as a property that holds only for c, i.e. $P_I = \{c\}$. Take any assignment of $\mathcal{L}$ in $\mathcal{M}$, i.e. $s : VAR \longrightarrow M$. Then $(\mathcal{M}, s) \models P(x)$ only when $s(x) = c$ for all $x \in VAR$.
$\mathcal{M} = [M, I]$ is a counter model for $(P(x) \Rightarrow \forall x P(x))$, as we found s such $(\mathcal{M}, s) \models P(x)$ and obviously $(\mathcal{M}, s) \not\models \forall x P(x)$. This proves that Deduction Theorem fails for $A(x)$ being an atomic formula $P(x)$.

The Fact 3 shows that the problem is with application of the generalization rule (G) to the formula $A \in \Gamma$. To handle this we introduce, after Mendelson(1987) the following notion.

Definition 22

Let A *be one of formulas in* Γ *and let*

$$B_1, B_2, ..., B_n \tag{34}$$

a deduction (proof of B_n *from* Γ*, together with* justification *at each step.*

We say that the formula B_i **depends upon** *A in the proof (34) if and only if*
(1) B_i is A and the justification *for B_i is $B_i \in \Gamma$*
or
(2) B_i is justified *as direct consequence by MP or (G) of some preceding formulas in the sequence (34), where at least one of these preceding formulas* **depends upon** *A.*

Here is a deduction

$$B_1, B_2, \ldots, B_5 \tag{35}$$

showing that

$$A, \ (\forall x A \Rightarrow C).$$

B_1 A, Hyp

B_2 $\forall x A$, $B_1, (G)$

B_3 $(\forall x A \Rightarrow C)$, Hyp

B_4 C, MP on B_2, B_3

B_5 $\forall x C$. (G)

Observe that the formulas A, C may, or may not have x as a free variable.

Example 2

In the derivation (35)
B_1 depends upon A,
B_2 depends upon A,
B_3 depends upon $(\forall x A \Rightarrow C)$,
B_4 depends upon A and $(\forall x A \Rightarrow C)$,
B_5 depends upon A and $(\forall x A \Rightarrow C)$.

Lemma 7

If B does not depend upon A in a deduction showing that $\Gamma, A \vdash B$, then $\Gamma \vdash B$.

Proof
Let $B_1, B_2, \ldots, B_n = B$ be a deduction of B from Γ, A in which B does not depend upon A. we prove by Induction that $\Gamma \vdash B$. Assume that Lemma 7 holds for all deductions of the length less than n. If $B \in \Gamma$ or $B \in LA$, then $\Gamma \vdash B$. If B is a direct consequence of two preceding formulas, then, since B does not depend upon A , neither do theses preceding formulas. By inductive hypothesis, theses preceding formulas have a proof from Γ alone. Hence so does B.

Now we are ready to formulate and prove the Deduction Theorem 11 for predicate logic.

Theorem 11 (Deduction Theorem)

For any formulas A, B of the language of proof system **H** *the following holds.*

(1) Assume that in some deduction showing that

$$\Gamma, A \ \vdash \ B,$$

no application *of the generalization rule (G) to a formula that depends upon A* has *as its quantified variable a free variable of A. Then*

$$\Gamma \ \vdash \ (A \Rightarrow B).$$

(2) If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$.

Proof
The proof extends the proof of the Deduction Theorem for propositional logic from chapter **??**. We adopt the propositional proof (for a different proof system) to the system **H** and adding the predicate case. For the sake of clarity and independence we write now the whole proof in all details.
(1) Assume that $\Gamma,\ A\ \vdash B$, i.e. that we have a formal proof

$$B_1, B_2, ..., B_n \tag{36}$$

of B from the set of formulas $\Gamma \cup \{A\}$. In order to prove that $\Gamma \vdash (A \Rightarrow B)$ we will prove the following a little bit stronger statement **S**.

$$\mathbf{S}: \quad \Gamma \vdash (A \Rightarrow B_i) \quad \text{for all } B_i \ \ (1 \le i \le n) \ \text{ in the proof (36) of } B.$$

Hence, in particular case, when $i = n$, we will obtain that also

$$\Gamma \vdash (A \Rightarrow B).$$

The proof of **S** is conducted by induction on i ($1 \le i \le n$).

Base Step $i = 1$.
When $i = 1$, it means that the formal proof (**??**) contains only one element B_1. By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that $B_1 \in LA$, or $B_1 \in \Gamma$, or $B_1 = A$, i.e.

$$B_1 \in LA \cup \Gamma \cup \{A\}.$$

Here we have two cases.

Case 1. $B_1 \in LA \cup \Gamma$.
Observe that the formula is a particular case of A2 of **H**. By assumption $B_1 \in LA \cup \Gamma$, hence we get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1 \ ; \ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

Case 2. $B_1 = A$.
When $B_1 = A$, then to prove $\Gamma \vdash (A \Rightarrow B)$ means to prove $\Gamma \vdash (A \Rightarrow A)$. But $(A \Rightarrow A) \in LA$ (axiom A21of **H**), i.e. $\vdash (A \Rightarrow A)$. By the monotonicity of the consequence we have that $\Gamma \vdash (A \Rightarrow A)$. The above cases conclude the proof of the Base case $i = 1$.

Inductive step
Assume that $\Gamma \vdash (A \Rightarrow B_k)$ for all $k < i$, we will show that using this fact we can conclude that also $\Gamma \vdash (A \Rightarrow B_i)$.

Consider a formula B_i in the sequence 36. By the definition, $B_i \in LA \cup \Gamma \cup \{A\}$ or B_i follows by MP from certain B_j, B_m such that $j < m < i$. We have to consider again two cases.

Case 1. $B_i \in LA \cup \Gamma \cup \{A\}$.
The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the Base Step for $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition.

Case 2. B_i is a conclusion of MP.
If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the sequence 36 such that $j < i, m < i,\ j \neq m$ and

$$(MP)\ \frac{B_j\ ;\ B_m}{B_i}.$$

By the inductive assumption, the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j) \tag{37}$$

and

$$\Gamma \vdash (A \Rightarrow B_m). \tag{38}$$

Moreover, by the definition of the Modus Ponens rule, the formula B_m has to have a form $(B_j \Rightarrow B_i)$, i.e. $B_m = (B_j \Rightarrow B_i)$, and the the inductive assumption (38) can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)), \quad for\ j < i. \tag{39}$$

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A3 of **H** and hence has a proof in **H**. By the monotonicity,

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))). \tag{40}$$

Applying the rule MP to formulas (40) and (39,) i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow (B_j \Rightarrow B_i))\ ;\ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)). \tag{41}$$

Applying again the rule MP to formulas 37 and 41, i.e. performing the following

$$(MP)\ \frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i).$$

Finally, suppose that there is some $j < i$ such that B_i is $\forall x B_j$. By hypothesis $\Gamma \vdash B_j$ and either (i) B_j does not depend upon A or (ii) x is not free variable in A.
We have two cases (i) and (ii) to consider.

(i) If B_j does not depend upon A , then by Lemma 7 $\Gamma \vdash B_j$ and, consequently, by the generalization rule (G), $\Gamma \vdash \forall x B_j$. Thus $\Gamma \vdash B_i$.

Now, by hypothesis $\Gamma \vdash B_j$ and by axiom A2, $\vdash (B_i \Rightarrow (A \Rightarrow B_i))$. Applying MP we get $\Gamma \vdash A \Rightarrow B_i)$.

(ii) If x is not free variable in A, then, by Completeness Theorem 6 and $\models (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$ we have that $\vdash (\forall x(A \Rightarrow B_j) \Rightarrow (A \Rightarrow \forall x B_j))$.

Since $\Gamma \vdash A \Rightarrow B_i)$, we get by the generalization rule (G), $\Gamma \vdash \forall x(A \Rightarrow B_j)$, and so, by MP, $\Gamma \vdash A \Rightarrow \forall x B_j)$; that is $\Gamma \vdash A \Rightarrow B_i)$.
This completes the induction and the case (1) holds for $i = n$.

(2) The proof of the implication

$$\text{if } \ \Gamma \vdash (A \Rightarrow B) \ \text{ then } \ \Gamma, A \vdash B$$

is straightforward. Assume that $\Gamma \vdash (A \Rightarrow B)$, hence by monotonicitywe have also that $\Gamma, A \ \vdash (A \Rightarrow B)$. Obviously, $\Gamma, A \ \vdash A$. Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$ i.e. we have proved that $\Gamma,\ A \vdash B$. This **ends the proof** of the Deduction Theorem for **H**.

3 Some other Axiomatizations

We present here some of most known, and historically important axiomatizations of classical predicate logic, i.e. the following Hilbert style proof systems.

1. Hilbert and Ackermann (1928)

D. Hilbert and W. Ackermann, *Grundzügen der Theoretischen Logik (Principles of Theoretical Logic)*, Springer - Verlag, 1928. The book grew from the courses on logic and foundations of mathematics Hilbert gave in years 1917-1922. He received help in writeup from Barnays and the material was put into the book by Ackermann and Hilbert. It was conceived as an introduction to mathematical logic and was followed by D. Hilbert and P. Bernays, *Grundzügen der Mathematik I,II.* Springer -Verlag, 1934, 1939.

Hilbert and Ackermann formulated and asked a question of the completeness for their deductive (proof) system. It was answered affirmatively by Kurt Gödel in 1929 with proof of his Completeness Theorem 6.

We define the Hilbert and Ackermann system **HA** following a pattern established for the **H** system (20). The original language use by Hilbert and Ackermann contained only negation $\neg$ and disjunction $\cup$ and so do we.

$$\mathbf{HA} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R} = \{(MP),\ (SB),\ (G1),\ (G2)\}), \tag{42}$$

where the set LA of logical axioms is as follows.

Propositional Axioms

A1 $(\neg(A \cup A) \cup A)$, A2 $(\neg A \cup (A \cup B))$,

A3 $(\neg(A \cup B) \cup (B \cup A))$, A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$,

for any $A, B, C, \in \mathcal{F}$.

Quantifiers Axioms

Q1 $(\neg \forall x A(x) \cup A(x))$, Q2 $(\neg A(x) \cup \exists x A(x))$,

Q2 $(\neg A(x) \cup \exists x A(x))$,

for any $A(x) \in \mathcal{F}$.

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule. It has, in the language $\mathcal{L}_{\{\neg,\cup\}}$, a form

$$(MP)\ \ \frac{A\ ;\ \ (\neg A \cup B)}{B}.$$

. (SB) is a **substitution rule**

$$(SB)\ \ \frac{A(x_1, x_2,\ \dots\ x_n)}{A(t_1, t_2,\ \dots\ t_n)},$$

where $A(x_1, x_2,\ \dots\ x_n) \in \mathcal{F}$ and $t_1, t_2,\ \dots\ t_n \in \mathbf{T}$.

$(G1), (G2)$ are **quantifiers generalization rules**.

$$(G1)\ \ \frac{(\neg B \cup A(x))}{(\neg B \cup \forall x A(x))}, \qquad (G2)\ \ \frac{(\neg A(x) \cup B)}{(\neg \exists x A(x) \cup B)},$$

where $A(x), B \in \mathcal{F}$ and B is such that x is not free in B.

The **HA** system is usually written now with the use of implication, i.e. as based on a language $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$, i.e. as a proof system

$$\mathbf{HAI} = (\mathcal{L}_{\{\neg, \Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R} = \{(MP),\ (SB),\ (G1),\ (G2)\}), \quad (43)$$

where the set LA of logical axioms is as follows.

Propositional Axioms

A1 $((A \cup A) \Rightarrow A)$, A2 $(A \Rightarrow (A \cup B))$,

A3 $((A \cup B) \Rightarrow (B \cup A))$, A4 $((\neg B \cup C) \Rightarrow ((A \cup B) \Rightarrow (A \cup C)))$,

for any $A, B, C, \in \mathcal{F}$.

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(x))$, Q2 $(A(x) \Rightarrow \exists x A(x))$,

for any $A(x) \in \mathcal{F}$.

Rules of Inference $\mathcal{R}$

(MP) is Modus Ponens rule

$$(MP)\ \ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

for any formulas $A, B \in \mathcal{F}$.

(SB) is a **substitution rule**

$$(SB)\ \ \frac{A(x_1, x_2,\ \dots\ x_n)}{A(t_1, t_2,\ \dots\ t_n)},$$

where $A(x_1, x_2,\ \dots\ x_n) \in \mathcal{F}$ and $t_1, t_2,\ \dots\ t_n \in \mathbf{T}$.

$(G1), (G2)$ are **quantifiers generalization rules**.

$$(G1)\ \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}, \qquad (G2)\ \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)},$$

where $A(x), B \in \mathcal{F}$ and B is such that x is not free in B.

The form of the quantifiers axioms Q1, Q2, and quantifiers generalization rule (Q2) is due to Bernays.

2. Mendelson (1987)

Here is the first order logic proof system **HM** as introduced in the Elliott Mendelson's book *Introduction to Mathematical Logic*, hence the name. (1987). It is an generalization to the predicate language of the proof system H_2 for propositional logic defined and studied in Chapter **??**.

$$\mathbf{HM} = (\mathcal{L}_{\{\neg,\cup\}}(\mathbf{P}, \mathbf{F}, \mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R} = \{(MP),\ (G)\}). \tag{44}$$

Propositional Axioms

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$,

for any $A, B, C, \in \mathcal{F}$.

Quantifiers Axioms

Q1 $(\forall x A(x) \Rightarrow A(t))$,

where where t is a term, A(t) is a result of substitution of t for all free occurrences of x in $A(x)$, and t is *free for* x in A(x), i.e. no occurrence of a variable in t becomes a bound occurrence in A(t).

Q2 $(\forall x(B \Rightarrow A(x)) \Rightarrow (B \Rightarrow \forall x A(x)))$, where $A(x), B \in \mathcal{F}$ and B is such that x is not free in B.

Rules of Inference $\mathcal{R}$

(MP) is the Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

for any formulas $A, B \in \mathcal{F}$.

(G) is the **generalization rule**

$$(G) \quad \frac{A(x)}{\forall x A(x)},$$

where $A(x) \in \mathcal{F}$ and $x \in VAR$.

Rasiowa-Sikorski (1950)

Rasiowa and Sikorski are the authors of the first algebraic proof of the Gödel completeness theorem ever given in 1950. Other algebraic proofs were later given by Rieger, Beth, Łos in 1951, and Scott in 1954.

Here is their original axiomatization.

$$RS = (\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P},\mathbf{F},\mathbf{C}),\ \mathcal{F},\ LA,\ \ \mathcal{R}). \tag{45}$$

Propositional Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

A12 $(A \cup \neg A)$,

for any $A, B, C \in \mathcal{F}$.

Rules of Inference $\mathcal{R} = \{(MP), (SB), (Q1), (Q2), (Q3), (Q4)\}$.

(MP) is **Modus Ponens** rule

$$(MP) \quad \frac{A\ ;\ (A \Rightarrow B)}{B},$$

for any formulas $A, B \in \mathcal{F}$.

(SB) is a **substitution rule**

$$(SB) \quad \frac{A(x_1, x_2, \ \ldots \ x_n)}{A(t_1, t_2, \ \ldots \ t_n)},$$

where $A(x_1, x_2, \ \ldots \ x_n) \in \mathcal{F}$ and $t_1, t_2, \ \ldots \ t_n \in \mathbf{T}$.

$(G1), (G2)$ are the following **quantifiers introduction rules**.

$$(G1) \quad \frac{(B \Rightarrow A(x))}{(B \Rightarrow \forall x A(x))}, \qquad (G2) \quad \frac{(A(x) \Rightarrow B)}{(\exists x A(x) \Rightarrow B)},$$

where $A(x), B \in \mathcal{F}$ and B is such that x is not free in B.

$(G3), (G3)$ are the following **quantifiers elimination rules**.

$$(G3) \quad \frac{(B \Rightarrow \forall x A(x))}{(B \Rightarrow A(x))}, \qquad (G4) \quad \frac{\exists x (A(x) \Rightarrow B)}{A(x) \Rightarrow B)},$$

where $A(x), B \in \mathcal{F}$ and B is such that x is not free in B.

The algebraic logic starts from purely logical considerations, abstracts from them, places them into a general algebraic contest, and makes use of other branches of mathematics such as topology, set theory, and functional analysis. For example, Rasiowa and Sikorski algebraic generalization of the completeness theorem for classical predicate logic is the following.

Theorem 12 (Rasiowa, Sikorski 1950)

For every formula A of the classical predicate calculus $S = \{\mathcal{L}, \mathcal{C}\}$ the following conditions are equivalent

i *A is derivable in RS;*

ii *A is valid in every realization of $\mathcal{L}$;*

iii *A is valid in every realization of $\mathcal{L}$ in any complete Boolean algebra;*

iv *A is valid in every realization of $\mathcal{L}$ in the field $B(X)$ of all subsets of any set $X \neq \emptyset$;*

v *A is valid in every semantic realization of $\mathcal{L}$ in any enumerable set;*

vi *there exists a non-degenerate Boolean algebra $\mathcal{A}$ and an infinite set J such that A is valid in every realization of $\mathcal{L}$ in J and $\mathcal{A}$;*

vii *$A_R(\mathbf{i}) = V$ for the canonical realization R of $\mathcal{L}$ in the Lindenbaum-Tarski algebra $\mathcal{LT}$ of S and the identity valuation $\mathbf{i}$;*

viii *A is a predicate tautology.*

3.1 Homework Problems

1. Prove that for any equality axioms (5) A and for every structure $\mathcal{M} = [M, I]$ and every $s : VAR \longrightarrow M$, $(\mathcal{M}, s) \models A$.

2. Let **H** be the proof system defined (20). Prove the following.

   (i) QA axioms Q1 and Q2 of **H** are predicate tautologies.

   (ii) The rules of inference (G), (G1), (G2) of **H** are sound.

3. A proof system S is *strongly sound* if for any rule of inference r of S, a conjunction of all premisses of r is logically equivalent with its conclusion.

   Show that the proof system **H** is not strongly sound.

4. Prove soundness theorem for Hilbert Ackerman system **HA** (42).

5. Given two proof systems S and K we say that S and K are equivalent and write it as $S \equiv K$ if they have the same sets of of theorems.
   Prove that $\mathbf{HA} \equiv \mathbf{HAI}$ for **HA** defined by (42) and **HAI** defined by (43)

6. We know that the Medndelson proof system **HM** defined by (44) is complete. Prove that $\mathbf{HM} \equiv \mathbf{H}$, where **H** be the proof system defined (20).

7. Let **RSE** be a proof system obtained from **RS** system defined by (45) by changing the language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}(\mathbf{P}, \mathbf{F}, \mathbf{C})$ of **RS** to the language with equality (definition 12) and adding Eguality Axioms (5) to the set LA of logocal axioms of **RS**. Prove Completeness Theorem 7 for **RSE**.

8. Prove Deduction Theorem 11 for Mendelson (19730 formalization.

9. In the proof of Deduction Theorem 11 for the proof system **H** we used gthe completeness of **H**. Write a proof of the Deduction Theorem 11 for **H** without use of its completeness.