CHAPTER 7

ch7 Introduction to Intuitionistic and Modal Logics

1 Introduction to Intuitionictic Logic

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as *intuitionism.* Intuitionism was originated by L. E. J. Brouwer in 1908. The first Hilbert style formalization of the intuitionistic logic, formulated as a proof system, is due to A. Heyting (1930). In this chapter we present a Hilbert style proof system I that is equivalent to the Heyting's original formalization and discuss the relationship between intuitionistic and classical logic.

There have been, of course, several successful attempts at creating semantics for the intuitionistic logic,i.e. to define formally a notion of the intuitionistic tautology. The most recent are Kripke models were defined by Kripke in 1964. The first intuitionistic semantics was defined in a form of pseudo- Boolean algebras by McKinsey, Tarski in 1944 - 1946. McKinsey, Tarski algebraic approach to the intuitionostic semantics (and classical) was followed by many authors and developed into a new field of *Algebraic Logic.* The pseudo- Boolean algebras are called also Heyting algebras.

An uniform presentation of algebraic models for classical, intuitionistic and modal logics was first given in a now classic algebraic logic book: *"Mathematics of Metamathematics"*, Rasiowa, Sikorski (1964).

The goal of this chapter is to give a presentation of the intuitionistic logic formulated as a proof system, discuss its algebraic semantics and the basic theorems that establish the relationship between classical and intuitionistic logics.

1.1 Philosophical Motivation

Intuitionists' view-point on the meaning of the basic logical and set theoretical concepts used in mathematics is different from that of most mathematicians in their research.

The basic difference lies in the interpretation of the word *exists.* For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence

$$\exists x A(x) \tag{1}$$ s

is true if it is a theorem of arithmetic, i.e. if it can be *deduced* from the axioms of arithmetic by means of classical logic. If a mathematician proves sentence (1), this does not mean that he is able to indicate a *method of construction* of a natural number n such that $A(n)$ holds.

For the intuitionist the sentence (1) is true only he is able to provide a constructive method of finding a number n such that $A(n)$ is true.

Moreover, the mathematician often obtains the proof of the existential sentence (1), i.e. of the sentence $\exists x A(x)$ by proving first a sentence

$$\neg\forall x\ \neg A(x). \qquad (2) \quad \boxed{\texttt{s1}}$$

Next he makes use of a classical tautology

$$(\neg\forall x\ \neg A(x)) \Rightarrow \exists x A(x)). \qquad (3) \quad \boxed{\texttt{s2}}$$

By applying Modus Ponens to (2) and (3) he obtains (1).

For the intuitionist such method is not acceptable, for it does not give any *method of constructing* a number n such that $A(n)$ holds. For this reason the intuitionist do not accept the classical tautology (3) i.e. $(\neg\forall x\ \neg A(x)) \Rightarrow \exists x A(x))$ as intuitionistic tautology, or as as an intuitionistically provable sentence.

Let us denote by $\vdash_I A$ and $\models_I A$ the fact that A is intuitionistically provable and that A is intuitionistic tautology, respectively. The proof system I for the intuitionistic logic has hence to be such that

$$\nvdash_I (\neg\forall x\ \neg A(x)) \Rightarrow \exists x A(x)).$$

The intuitionistic semantics I has to be such that one can prove in that also

$$\not\models_I (\neg\forall x\ \neg A(x)) \Rightarrow \exists x A(x)).$$

The above means also that intuitionists interpret differently the meaning of propositional connectives.

Intuitionistic implication

The intuitionistic implication $(A \Rightarrow B)$ is considered by to be true if there exists a method by which a *proof of* B can be deduced from the proof of A. In the case of the implication

$$(\neg\forall x\ \neg A(x)) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence $(\neg\forall x\ \neg A(x))$, permits is to obtain an intuitionistic proof of the sentence $\exists x A(x)$, i.e. to construct a number n such that $A(n)$ holds, hence we can't accept it as an intuitionistic theorem or tautology.

Intuitionistic negation

The negation and the disjunction are also understood differently. The sentence $\neg A$ is considered intuitionistically true if the acceptance of the sentence A leads to absurdity.

As a result of above understanding of negation and implication we have that in the intuitionistic logic I

$$\vdash_I (A \Rightarrow \neg\neg A)$$

but

$$\nvdash_I (\neg\neg A \Rightarrow A).$$

Consequently, the intuitionistic semantics I has to be such that

$$\models_I (A \Rightarrow \neg\neg A)$$

and

$$\not\models_I (\neg\neg A \Rightarrow A).$$

Intuitionistic disjunction

The intuitionist regards a disjunction $(A \cup B)$ as true if one of the sentences A, B is true and there is a method by which it is possible to find out which of them is true. As a consequence a classical law of excluded middle

$$(A \cup \neg A)$$

is not acceptable by the intuitionists since there is no general method of finding out, for any given sentence A, whether A or $\neg A$ is true. This means that the intuitionistic logic must be such that

$$\nvdash_I (A \cup \neg A)$$

and the intuitionistic semantics I has to be such that

$$\not\models_I (A \cup \neg A).$$

Intuitionists' view of the concept of infinite set also differs from that which is generally accepted in mathematics. Intuitionists reject the idea of infinite set as a closed whole. They look upon an infinite set as something which is constantly in a state of formation. Thus, for example, the set of all natural numbers is infinite in the sense that to any given finite set of natural numbers it is always possible to add one more natural number. The notion of the set of all subsets of the set of all natural numbers is not regarded meaningful. Thus intuitionists reject the general idea of a set as defined by a modern set theory.

An exact exposition of the basic ideas of intuitionism is outside the range of our investigations. Our goal is to give a presentation of of the intuitionistic logic, which is a sort of reflection of intuitionistic ideas formulated as a proof system.

1.2 Algebraic Intuitionistic Semantics and Completeness Theorem

There are many proof systems describing the intuitionistic logic. We define now a system I with a set of axioms that is due to Rasiowa (1959). We adopted this axiomatization for two reasons. Firs is that it is the most natural and appropriate set of axioms to carry the the algebraic proof of the completeness theorem and the second is that they visibly describe the main difference between intuitionistic and classical logic. Namely, by adding the only one more axiom $(A \cup \neg A)$ we get a (complete) formalization for classical logic. Here are the components if the proof system I.

Language We adopt a propositional language $\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ with the set of formulas denoted by $\mathcal{F}$.

Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

where A, B, C are any formulas in $\mathcal{L}$.

Rules of inference

We adopt the Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only inference rule.

A proof system

$$\mathbf{I} = (\ \mathcal{L}, \mathcal{F}\ A1 - A11,\ (MP)\ ), \tag{4}$$ Isys

$A1 - A11$ defined above, is called a Hilbert style formalization for Intuitionistic propositional logic.

We introduce, as usual, the notion of a formal proof in I and denote by

$$\vdash_I A$$

the fact that A has a formal proof in I, or that that A is *intuitionistically provable* in I.

1.3 Algebraic Semantics and Completeness Theorem

We shortly present here Tarski, Rasiowa, and Sikorski psedo-Boolean algebra semantics and discuss the algebraic completeness theorem for the intuitionistic propositional logic.

We leave the Kripke semantics for the reader to explore from other, multiple sources.

Here are some basic definitions.

Here are some basic definitions.

Relatively Pseudo-Complemented Lattice
A lattice $(B, \cap, \cup)$ is said to be *relatively pseudo-complemented* (Birkhoff, 1935) if for any elements $a, b \in B$, there exists the greatest element c, such that $a \cap c \leq b$. Such element is denoted by $a \Rightarrow b$ and called the *pseudo-complement of a relative to b*. By definition

$$x \leq a \Rightarrow b \ \text{ if and only if } \ a \cap x \leq b \text{ for all } x, a, b \in B. \tag{5}$$ p-c

The equation (5) can serve as the definition of the **relative pseudo-complement** $a \Rightarrow b$.

unit1 **Fact 1** *Every relatively pseudo-complemented lattice* $(B, \cap, \cup)$ *has the greatest element, called a unit element and denoted by 1.*

Proof Observe that $a \cap x \leq a$ for all $x, a \in B$. By (5) we have that $x \leq a \Rightarrow a$ for all $x \in B$, i.e. $a \Rightarrow a = 1$.

An abstract algebra

$$\mathcal{B} = (B,\ 1,\ \Rightarrow,\ \cap,\ \cup, \Rightarrow) \tag{6}$$ rpc

is said to be a **relatively pseudo-complemented lattice** if $(B, \cap, \cup)$ is *relatively pseudo-complemented lattice* with the *relative pseudo-complement* $\Rightarrow$ defined by (5) and with the unit element 1 (Fact 1).

Relatively Pseudo-complemented Set Lattices

Consider a topological space X with an interior operation I. Let $\mathcal{G}(X)$ be the class of all open subsets of X and $\mathcal{G}^*(X)$ be the class of all both dense and open subsets of X. Then the algebras

$$(\mathcal{G}(X),\ X,\ \cup,\ \cap, \Rightarrow), \quad (\mathcal{G}^*(X),\ X,\ \cup,\ \cap, \Rightarrow),$$

where $\cup$, $\cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

are relatively pseudo-complemented lattices.
Clearly, all sub algebras of these algebras are also relatively pseudo-complemented lattices, called *relatively pseudo-complemented set lattices.* They are typical examples of relatively pseudo-complemented lattices

Pseudo - Boolean Algebra (Heyting Algebra)
An algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg), \tag{7}$$ `Pb-alg`

is said to be a *pseudo - Boolean algebra* if and only if $(B,\ 1,\ \Rightarrow,\ \ \cap,\ \cup)$ it is a relatively pseudo-complemented lattice (6) in which a zero element 0 exists and $\neg$ is a one argument operation defined as follows

$$\neg a = a \Rightarrow 0 \tag{8}$$ `i-neg`

The operation $\neg$ is called a **pseudo-complementation**.

The pseudo - Boolean algebras are also called **Heyting algebras** to stress their connection to the intuitionistic logic.

Let X be topological space with an interior operation I. Let $\mathcal{G}(X)$ be the class of all open subsets of X. Then

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg), \tag{9}$$ `p-f`

where $\cup$, $\cap$ are set-theoretical operations of union, intersection, $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

and $\neg$ is defined as

$$\neg Y = Y \Rightarrow \emptyset = I(X - Y), \quad \text{for all } Y \subseteq X$$

is a pseudo - Boolean algebra.
Every sub algebra of $\mathcal{G}(X)$ is also a pseudo-Boolean algebra. They are called **pseudo-fields of sets.**

The following Theorem 1 states that pseudo-fields are typical examples of pseudo - Boolean algebras. The theorems of this type are often called **Stone Representation Theorems** to remember an American mathematician H.M Stone. He was one of the first to initiate the investigations between logic and general topology in the article *"The Theory of Representations for Boolean Algebras"*,Trans. of the Amer.Math, Soc 40, 1036.

thm:repr **Theorem 1 (Representation Theorem)** *(McKinsey, Tarski, 1946)*

For every pseudo - Boolean (Heyting) algebra (7)

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg),$$

there exists a monomorphism h of $\mathcal{B}$ into a pseudo-field (9) $\mathcal{G}(X)$ of all open subsets of a compact topological T_0 space X.

Another typical (and interesting) example of a class of pseudo - Boolean algebras is the following.

Linear Pseudo - Boolean Algebra

Let $(B, \leq)$ be a chain (linearly ordered set) with the greatest element 1 and the least element (smallest) 0.
An algebra

$$\mathcal{H} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg) \tag{10}$$ Lpb

is called a *linear pseudo - Boolean algebra* if and only if its operations are defined as follows.

For any $a, b \in B$,

$$a \cup b = max\{a,\ b\}, \qquad a \cap b = min\{a,\ b\},$$

$$a \Rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise,} \end{cases} \tag{11}$$ L-impl

and define the *pseudo-complementation* $\neg$ as

$$\neg a = a \Rightarrow 0.$$

We leave the proof that (10) is a pseudo-Boolean algebra as a homework exercise. Observe that the linear pseudo - Boolean algebra (10) is a generalization of the 3-valued Heyting semantics defined in chapter **??**.

Algebraic Models

We say that a formula A is an **intuitionistoc tautology** if and only if any pseudo-Boolean algebra (7) is a *model* for A. This kind of models because their connection to abstract algebras are called **algebraic models**.

We put it formally as follows.

`I-model` **Definition 1 (Intuitionistic Algebraic Model)**

Let A be a formula of the language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ and let $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup,\neg)$ be a pseudo - Boolean topological Boolean algebra (7).

We say that the algebra $\mathcal{B}$ is a **model** *for the formula A and denote it by*

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$.

`I-taut` **Definition 2 (Intuitionistic Tautology)**

The formula A is an **intuitionistic tautology** *and is denoted by*

$$\models_I A \quad \textit{if and only if}\ \mathcal{B} \models A$$

for all pseudo-Boolean algebras $\mathcal{B}$.

In Algebraic Logic the notion of tautology is often defined using a notion "a formula A is **valid** in an algebra $\mathcal{B}$ ". It is formally defined as follows.

`I-valid` **Definition 3**

A formula A **is valid** *in a pseudo-Boolean algebra $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup,\neg)$, if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$.*

Directly from definition 2 and definition 3 we get the following.

`Ivalid` **Fact 2** *For any formula A, $\models_I A$ if and only if A* **is valid** *in all pseudo-Boolean algebras.*

We write now $\vdash_I A$ to denote any proof system for the Intuitionistic propositional logic, and in particular the Hilbert style formalization for Intuitionistic propositional logic I defined by (4).

`I-com` **Theorem 2 (Intuitionistic Completeness Theorem)** *(Mostowski 1948)*

For any formula A of $\mathcal{L}_{\{\cup,\ \cap,\Rightarrow,\neg\}}$,

$$\vdash_I A \quad \textit{if and only if} \quad \models_I A.$$

The intuitionistic completeness theorem 2 follows also directly from the following general algebraic completeness theorem 3 that combines results of of Mostowski (1958), Rasiowa (1951) and Rasiowa-Sikorski (1957).

I-thm **Theorem 3 (Algebraic Completeness Theorem)**

For any formula A of $\mathcal{L}= \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ the following conditions are equivalent.

(i) $\vdash_I A$,

(ii) $\models_I A$,

(iii) *A is valid in every pseudo-Boolean algebra*

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$$

of open subsets of any topological space X,

(iv) *A is valid in every pseudo-Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A.*

Moreover, each of the conditions (i) - (iv) *is equivalent to the following one.*

(v) *A is valid in the pseudo-Boolean algebra $(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$ of open subsets of a dense-in -itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space X).*

The following theorem follows from the equivalence of conditions (i) and (iv).

Idec **Theorem 4 (Decidability)**

Every proof system for the intuitionistic propositional logic is decidable.

Examples of intuitionistic propositional tautologies

The following classical tautologies are provable in I and hence are also intuitionistic tautologies.

$$(A \Rightarrow A), \tag{12}$$ i1

$$(A \Rightarrow (B \Rightarrow A)), \tag{13}$$ i2

$$(A \Rightarrow (B \Rightarrow (A \cap B))), \tag{14}$$ i3

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))), \tag{15}$$ i4

$$(A \Rightarrow \neg\neg A), \tag{16}$$ i5

$$\neg(A \cap \neg A), \tag{17}$$ i6

$$((\neg A \cup B) \Rightarrow (A \Rightarrow B)), \tag{18}$$ i7

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), \tag{19}$$ i8

$$((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B)), \tag{20}$$ i9

$$((\neg A \cup \neg B) \Rightarrow (\neg A \cap \neg B)), \tag{21}$$ i10

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)), \tag{22}$$ i11

$$((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)), \quad (23) \quad \boxed{\texttt{i12}}$$

$$(\neg\neg\neg A \Rightarrow \neg A), \quad (24) \quad \boxed{\texttt{i13}}$$

$$(\neg A \Rightarrow \neg\neg\neg A), \quad (25) \quad \boxed{\texttt{i14}}$$

$$(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B)), \quad (26) \quad \boxed{\texttt{i15}}$$

$$((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B)), \quad (27) \quad \boxed{\texttt{i16}}$$

Examples of classical tautologies that are not intuitionistic tautologies

The following classical tautologies are not intuitionistic tautologies.

$$(A \cup \neg A), \quad (28) \quad \boxed{\texttt{ni1}}$$

$$(\neg\neg A \Rightarrow A), \quad (29) \quad \boxed{\texttt{ni2}}$$

$$((A \Rightarrow B) \Rightarrow (\neg A \cup B)), \quad (30) \quad \boxed{\texttt{ni3}}$$

$$(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B)), \quad (31) \quad \boxed{\texttt{ni4}}$$

$$((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A)), \quad (32) \quad \boxed{\texttt{ni5}}$$

$$((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)), \quad (33) \quad \boxed{\texttt{ni6}}$$

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A), \quad (34) \quad \boxed{\texttt{ni7}}$$

1.4 Connection Between Classical and Intuitionistic Tautologies

The intuitionistic logic has been created as a rival to the classical one. So a question about the relationship between these two is a natural one. We present here some examples of tautologies and some historic results about the connection between the classical and intuitionistic logic.

The first connection is quite obvious. It was proved by Rasiowa and Sikorski in 1964 that by adding the axiom

A12 $(A \cup \neg A)$

to the set of axioms of our system I defined by (4) we obtain a Hilbert proof system H that is **complete** with respect to classical semantics.

This proves the following.

it1 **Theorem 5**

Every formula that is derivable intuitionistically is classically derivable, i.e.

$$\textit{if} \quad \vdash_I A, \quad \textit{then} \quad \vdash A,$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

We write

$$\models \ A \quad \text{and]} \ \models_I A$$

to denote that A is a classical and intuitionistic tautology, respectively.

As both proof systems, I and H are complete under respective semantics, we can state this as the following relationship between classical and intuitionistic tautologies.

it2 **Theorem 6**

For any formula $A \in \mathcal{F}$,

$$\textit{if} \quad \models_I A, \quad \textit{then} \quad \models A.$$

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa. It has been proved by Glivenko in 1929 and independently by Tarski in 1938.

it3 **Theorem 7 (Glivenko, Tarski)**

For any formula $A \in \mathcal{F}$, A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable, i.e.

$$\vdash_I A \quad \textit{if and only if} \quad \vdash \neg\neg A$$

where we use symbol $\vdash$ for classical provability.

it4 **Theorem 8 (McKinsey, Tarski, 1946)**

For any formula $A \in \mathcal{F}$, A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models \ A \ \ if\ and\ only\ if \ \ \models_I \neg\neg A.$$

The following relationships were proved by Gödel.

it5 **Theorem 9 (Gödel, 1931)**

For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is an intuitionistically provable, i.e.

$$\vdash\ (A \Rightarrow \neg B)\ \ if\ and\ only\ if\ \ \vdash_I (A \Rightarrow \neg B).$$

it6 **Theorem 10 (Gödel, 1931)**

If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable.

By the completeness of classical and intuitionisctic logics we get the following equivalent semantic form of theorems 9 and 10.

it7 **Theorem 11**

A formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models\ (A \Rightarrow \neg B)\ \ if\ and\ only\ if\ \ \models_I (A \Rightarrow \neg B).$$

it8 **Theorem 12**

If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology.

On intuitionistically derivable disjunction

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when neither A nor B is a tautology. The tautology $(A \cup \neg A)$ is the simplest example. This does not hold for the intuitionistic logic.

This fact was stated without the proof by Gödel in 1931 and proved by Gentzen in 1935 via his proof system **LI** which is presented in chapter **??** and discussed in the next section 2.

it9 **Theorem 13 (Gentzen 1935)**

A disjunction $(A \cup B)$ is intuitionistically provable if and only if either A or B is intuitionistically provable , i.e.

$$\vdash_I (A \cup B)\ \ \ if\ and\ only\ if\ \ \vdash_I A\ \ or\ \ \vdash_I B.$$

We obtain, via the Completeness Theorem 2 the following equivalent semantic version of the above.

it10 **Theorem 14**

A disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad if\ and\ only\ if \quad \models_I A \ \ or \ \ \models_I B.$$

2 Gentzen Sequent System LI

G-LI

In 1935 G. Gentzen formulated a first syntactically decidable formalizations for classical and intuitionistic logic and proved its equivalence with the Heyting's original Hilbert style formalization. He named his classical system **LK** (K for Klassisch) and intuitionistic system **LI** (I for Intuitionistisch). In order to prove the completeness of the system **LK** and proving the adequacy of **LI** he introduced a special rule, called *cut rule* that corresponds to the Modus Ponens rule in Hilbert proof systems. Then, as the next step he proved the now famous Gentzen *Hauptzatz*, called in English the *Cut Elimination Theorem*.

The Gentzen original proof system **LI** is a particular case of his proof system **LK** for the classical logic. Both of them were presented in chapter **??** together with the proof of the *Hauptzatz* for both, **LK** and **LI** systems.

The elimination of the cut rule and the structure of other rules makes it possible to define effective automatic procedures for proof search, what is impossible in a case of the Hilbert style systems.

The Gentzen system **LI** is defined as follows.

Language of LI

Let $SQ = \{\ \Gamma \longrightarrow \Delta :\ \Gamma, \Delta \in \mathcal{F}^*\ \}$ be the set of all Gentzen sequents built out of the formulas of the language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}} \tag{35}$$ I-lan

and the additional symbol $\longrightarrow$.

In order to describe the the intuitionistic logic we deal, after Gentzen, only with sequents of the form $\Gamma \ \longrightarrow \ \Delta$, where Δ consists of at most one formula. I.e. we assume that all **LI** sequents are elements of a following subset ISQ of the set SQ of all sequents.

$$ISQ = \{\Gamma \ \longrightarrow \ \Delta : \quad \Delta \ \text{ consists of at most one formula }\}. \tag{36}$$ iseq

The set ISQ is called the set of all intuitionistic sequents; the **LI** sequents.

Axioms of LI

As the axioms of **LI** we adopt any sequent from the set ISQ defined by (36), which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma_1, A, \Gamma_2 \;\longrightarrow\; A, \tag{37}$$

iax

for any formula $A \in \mathcal{F}$ of the language (35) and for any sequences $\Gamma_1, \Gamma_2 \in \mathcal{F}^*$.

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules. They are defined as follows.

Structural Rules of LI

Weakening

$$(\rightarrow weak)\ \frac{\Gamma \;\longrightarrow}{\Gamma \;\longrightarrow\; A}\ .$$

A is called the weakening formula.

Contraction

$$(contr \rightarrow)\ \frac{A, A, \Gamma \;\longrightarrow\; \Delta}{A, \Gamma \;\longrightarrow\; \Delta},$$

A is called the contraction formula , Δ contains at most one formula.

Exchange

$$(exchange \rightarrow)\ \frac{\Gamma_1, A, B, \Gamma_2 \;\longrightarrow\; \Delta}{\Gamma_1, B, A, \Gamma_2 \;\longrightarrow\; \Delta},$$

Δ contains at most one formula.

Logical Rules of LI

Conjunction rules

$$(\cap \rightarrow)\ \frac{A, B, \Gamma \;\longrightarrow\; \Delta}{(A \cap B), \Gamma \;\longrightarrow\; \Delta}, \qquad (\rightarrow \cap)\ \frac{\Gamma \;\longrightarrow\; A\ ;\ \Gamma \;\longrightarrow\; B}{\Gamma \;\longrightarrow\; (A \cap B)},$$

Δ contains at most one formula.

Disjunction rules

$$(\rightarrow \cup)_1\ \frac{\Gamma \;\longrightarrow\; A}{\Gamma \;\longrightarrow\; (A \cup B)}, \qquad (\rightarrow \cup)_2\ \frac{\Gamma \;\longrightarrow\; B}{\Gamma \;\longrightarrow\; (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \; ; \; B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}, \qquad (\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \; ; \; B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta},$$

Δ contains at most one formula.

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow}, \qquad (\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}.$$

Formally we define:

$$\mathbf{LI} = (\mathcal{L}, \; ISQ, \; LA, \; \text{Structural rules}, \; \text{Logical rules}), \tag{38}$$ LI

where ISQ is defined by (36), Structural rules and Logical rules are the inference rules defined above, and LA is the axiom defined by the schema (37).

We write

$$\vdash_{LI} \Gamma \longrightarrow \Delta$$

to denote that the sequent $\Gamma \longrightarrow \Delta$ has a proof in **LI**.

We say that a formula $A \in \mathcal{F}$ has a proof in **LI** and write it as

$$\vdash_{LI} A$$

when the sequent $\longrightarrow A$ has a proof in **LI**, i.e.

$$\vdash_{LI} A \quad \text{if and only if} \quad \vdash_{LI} \longrightarrow A.$$

The completeness of of our cut-free **LI** follows directly from LI Hauptzatz Theorem proved in chapter **??** and the Intuitionistic Completeness Theorem 2. The proof is a straightforward adaptation of the proof of cut free **LK** Completeness Theorem proved in chapter **??** and is left as a homework exercise.

LI-compl **Theorem 15 (Completeness of LI)**

For any sequent $\Gamma \longrightarrow \Delta \in ISQ$*,*

$$\vdash_{LI} \Gamma \longrightarrow \Delta \quad \textit{if and only of} \quad \models_I \Gamma \longrightarrow \Delta.$$

In particular, for any formula A,

$$\vdash_{LI} A \quad \textit{if and only of} \quad \models_I A.$$

G-disj **Theorem 16 (Intuitionistically Derivable Disjunction)**

For any formulas A, B,

$$\vdash_{LI} (A \cup B) \quad \textit{if and only if} \quad \vdash_{LI} A \quad \textit{or} \quad \vdash_{LI} B.$$

In particular, a disjunction $(A \cup B)$ is intuitionistically provable in any proof system I if and only if either A or B is intuitionistically provable in I.

The particular form the theorem 16 was stated without the proof by Gödel in 1931. The theorem proved by Gentzen in 1935 via his Hauptzatz Theorem.

Proof

Assume $\vdash_{LI} (A \cup B)$. This equivalent to $\vdash_{LI} \longrightarrow (A \cup B)$. The last step in the proof of $\longrightarrow (A \cup B)$i **LI** must be the application of the rule $(\rightarrow \cup)_1$ to the sequent $\longrightarrow A$, or the application of the rule $(\rightarrow \cup)_2$ to the sequent $\longrightarrow B$. There is no other possibilities. We have proved that $\vdash_{LI} (A \cup B)$ implies $\vdash_{LI} A$ or $\vdash_{LI} B$. The inverse is obvious by respective applications of rules $(\rightarrow \cup)_1$ $(\rightarrow \cup)_2$ to $\longrightarrow A$ and $\longrightarrow B$.

2.1 Decomposition Trees in LI

sec:dtrees

Search for proofs in **LI** is a much more complicated process then the one in classical systems **RS** or **GL** defined in chapter **??**.

Here, as in any other Gentzen style proof system, proof search procedure consists of building the decomposition trees.

In **RS** the decomposition tree $\mathbf{T}_A$ of any formula A, and hence of any sequence Γ is always unique.

In **GL** the "blind search" defines, for any formula A a finite number of decomposition trees, but it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules.

In **LI** the structural rules play a vital role in the proof construction and hence, in the proof search. We consider here a number of examples to show the complexity of the problem of examining possible decomposition trees for a given formula A. We are going to see that the fact that a given decomposition tree ends with an axiom leaf does not always imply that the proof does not exist. It might only imply that our search strategy was not good. Hence the problem of deciding whether a given formula A does, or does not have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic.

Before we define a heuristic method of searching for proof and deciding whether such a proof exists or not in **LI** we make some observations.

Observation 1: the logical rules of **LI** are similar to those in Gentzen type classical formalizations we examined in previous chapters in a sense that each of them introduces a logical connective.

Observation 2: The process of searching for a proof is, as before a decomposition process in which we use the inverse of logical and structural rules as decomposition rules.

For example the implication rule:

$$(\rightarrow\Rightarrow)\ \ \frac{A, \Gamma \ \longrightarrow\ B}{\Gamma \ \longrightarrow\ (A \Rightarrow B)}$$

becomes an **implication decomposition rule** (we use the same name $(\rightarrow\Rightarrow)$ in both cases)

$$(\rightarrow\Rightarrow)\ \ \frac{\Gamma \ \longrightarrow\ (A \Rightarrow B)}{A, \Gamma \ \longrightarrow\ B}.$$

Observation 3: we write our proofs in as trees, instead of sequences of expressions, so the proof search process is a process of building a decomposition tree. To facilitate the process we write, as before, the decomposition rules, structural rules included in a "tree " form.

For example the the above implication decomposition rule is written as follows.

$$\Gamma \ \longrightarrow\ (A \Rightarrow B)$$

$$\mid (\rightarrow\Rightarrow)$$

$$A, \Gamma \ \longrightarrow\ B$$

The two premisses implication rule $(\Rightarrow\rightarrow)$ written as the tree decomposition rule becomes

$$(A \Rightarrow B), \Gamma \ \longrightarrow$$

$$\bigwedge(\Rightarrow\rightarrow)$$

$$\Gamma \ \longrightarrow\ A \qquad\qquad B, \Gamma \ \longrightarrow$$

For example the structural weakening rule is written as the decomposition rule is written as

$$(\rightarrow weak)\ \frac{\Gamma\ \longrightarrow\ A}{\Gamma\ \longrightarrow}$$

We write it in a tree form as follows.

$$\Gamma\ \longrightarrow\ A$$

$$\mid (\rightarrow weak)$$

$$\Gamma\ \longrightarrow$$

We define, as before the notion of decomposable and indecomposable formulas and sequents as follows.

Decomposable formula is any formula of the degree ≥ 1.

Decomposable sequent is any sequent that contains a decomposable formula.

Indecomposable formula is any formula of the degree 0, i.e. any propositional variable.

Remark: In a case of formulas written with use of capital letters A, B, C, .. etc, we treat these letters as propositional variables, i.e. as *indecomposable formulas.*

Indecomposable sequent is a sequent formed from indecomposable formulas only.

Decomposition tree construction (1): given a formula A we construct its decomposition tree $\mathbf{T_A}$ as follows.

Root of the tree is the sequent $\longrightarrow A$.

Given a node n of the tree we identify a decomposition rule applicable at this node and write its premisses as the leaves of the node n.

We stop the decomposition process when we obtain an axiom or all leaves of the tree are indecomposable.

Observation 4: the decomposition tree $\mathbf{T_A}$ obtained by the construction (1) most often is not unique.

Observation 5: the fact that we find a decomposition tree $\mathbf{T_A}$ with non-axiom leaf does not mean that $\not\vdash_{LI} A$. This is due to the role of structural rules in **LI** and will be discussed later in the chapter.

We illustrate the problems arising with proof search procedures, i.e. decomposition trees construction in the next section 2.2 and give a heuristic proof searching procedure in the section 2.3.

sec: examles

2.2 Proof Search Examples

We perform proof search and decide the existence of proofs in **LI** for a given formula $A \in \mathcal{F}$ by constructing its decomposition trees $\mathbf{T_A}$. We examine here some examples to show the complexity of the problem.

Remark
In the following and similar examples when building the decomposition trees for formulas representing general schemas we treat the capital letters $A, B, C, D...$ as propositional variables, i.e. as *indecomposable formulas.*

Example 1

Determine whether $\vdash_{\mathbf{LI}} \longrightarrow ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$.

This means that we have to construct some, or all decomposition trees of

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

If we find a decomposition tree such that all its leaves are axioms, we have a proof.

If all possible decomposition trees have a non-axiom leaf, proof of A in **LI** does not exist.

Consider the following decomposition tree of $\longrightarrow ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$.

T1

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge (\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad\qquad B \longrightarrow B$$

$$non - axiom \qquad\qquad\qquad axiom$$

The tree **T1** has a non-axiom leaf, so it does not constitute a proof in **LI**. But this fact does not yet prove that proof doesn't exist, as the decomposition tree in **LI** is not always unique.

Let's consider now the following tree.

T2

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$$\begin{array}{cc}
A, \neg A, \neg B \longrightarrow & B, \neg A, \neg B \longrightarrow \\
\mid (exch \longrightarrow) & \mid (exch \longrightarrow) \\
\neg A, A, \neg B \longrightarrow & B, \neg B, \neg A \longrightarrow \\
\mid (\neg \longrightarrow) & \mid (exch \longrightarrow) \\
A, \neg B \longrightarrow A & \neg B, B, \neg A \longrightarrow \\
axiom & \mid (\neg \longrightarrow) \\
 & B, \neg A \longrightarrow B \\
 & axiom
\end{array}$$

All leaves of **T2** are axioms, what proves that **T2** is a proof of A and hence we proved that

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

Example 2

Part 1: Prove that

$$\vdash_{\mathbf{LI}} \longrightarrow (A \Rightarrow \neg\neg A),$$

Part 2: Prove that

$$\nvdash_{\mathbf{LI}} \longrightarrow (\neg\neg A \Rightarrow A).$$

Solution of Part 1

To prove that

$$\vdash_{\mathbf{LI}} \longrightarrow (A \Rightarrow \neg\neg A)$$

we have to construct some, or all decomposition trees of

$$\longrightarrow (A \Rightarrow \neg\neg A).$$

We treat the sub formulas A, B as *indecomposable formulas.*

Consider the following decomposition tree.

$$\begin{array}{c}
\mathbf{T} \\
\longrightarrow (A \Rightarrow \neg\neg A). \\
\mid (\longrightarrow \Rightarrow) \\
A \longrightarrow \neg\neg A \\
\mid (\longrightarrow \neg) \\
\neg A, A \longrightarrow
\end{array}$$

$$| (\neg \longrightarrow)$$

$$A \longrightarrow A$$

$$axiom$$

All leaves of **T** are axioms what proves that **T** is a proof of $\longrightarrow (A \Rightarrow \neg\neg A)$ and we don't need to construct other decomposition trees.

Solution of Part 2
To prove that

$$\nvdash_{\mathbf{LI}} \; \longrightarrow (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of $(A \Rightarrow \neg\neg A)$ and show that each of them has an non-axiom leaf.

Consider the first decomposition tree defined as follows.

T1

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$first\ of\ 2\ choices : (\rightarrow\Rightarrow), (\rightarrow weak)$$

$$| (\rightarrow\Rightarrow)$$

$$\neg\neg A \longrightarrow A$$

$$first\ of\ 2\ choices : (\rightarrow weak), (contr \rightarrow)$$

$$| (\rightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$first\ of\ 2\ choices : (\neg \rightarrow), (contr \rightarrow)$$

$$| (\neg \rightarrow)$$

$$\longrightarrow \neg A$$

$$first\ of\ 2\ choices : (\neg \rightarrow), (\rightarrow weak)$$

$$| (\rightarrow \neg)$$

$$A \longrightarrow$$

$$indecomposable$$

$$non-axiom$$

We use the first tree created to define all other possible decomposition trees by exploring the alternative search paths as indicated at the nodes of the tree.

T1

$\longrightarrow (\neg\neg A \Rightarrow A)$

$\mid (\longrightarrow \Rightarrow)$

one of 2 choices

$\neg\neg A \longrightarrow A$

]

$\mid (contr \longrightarrow)$

second of 2 choices

$\neg\neg A, \neg\neg A \longrightarrow A$

$\mid (\longrightarrow weak)$

first of 2 choices

$\neg\neg A, \neg\neg A \longrightarrow$

$\mid (\neg \longrightarrow)$

first of 2 choices

$\neg\neg A \longrightarrow \neg A$

$\mid (\longrightarrow \neg)$

the only choice

$A, \neg\neg A \longrightarrow$

$\mid (exch \longrightarrow)$

the only choice

$\neg\neg A, A \longrightarrow$

$\mid (\longrightarrow \neg)$

the only choice

$A \longrightarrow \neg A$

$\mid (\longrightarrow \neg)$

first of 2 choices

$A, A \longrightarrow$

indecomposable

non − axiom

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more complicated trees, due to the presence of structural rules. Observe that the "blind" application of ($contr \longrightarrow$) gives an infinite number of decomposition trees. To decide that none of them will produce a proof we need some extra knowledge about patterns of their construction, or just simply about the number useful of application of structural rules within the proofs.

In this case we can just make an "external" observation that the our first tree **T1** is in a sense a minimal one; that all other trees would only complicate this one in an inessential way, i.e. we will never produce a tree with all axioms leaves.

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its correctness require some extra knowledge. We are going to discuss a motivation and an heuristics for the proof search in the next section.

Within the scope of this book we accept the "external" explanation for the heuristics we use as a sufficient solution.

As we can see from the above examples structural rules and especially the ($contr \rightarrow$) rule complicates the proof searching task.

The Gentzen type proof systems **RS** and **GL** from chapter don't contain the structural rules and are complete with respect to classical semantics, as is the original Gentzen system **LK**, which does contain the structural rules. As (via Completeness Theorem) all three classical proof system **RS, GL, LK** are equivalent we can say that the structural rules can be eliminated from the system **LK**.

A natural question of elimination of structural rules from the intutionistic Gentzen system **LI** arizes.

The following example illustrates the negative answer.

Example 3

We know, by the theorem about the connection between classical and intuitionistic logic (theorem 6) and corresponding Completeness Theorems that for any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_I \neg\neg A,$$

where $\models A$ means that A is a classical tautology, $\vdash_I$ means that A is intutionistically provable, i.e. is provable in any intuitionistically complete proof system. The system **LI** is intuitionistically complete, so we have that for any formula A,

$$\models A \quad \text{if and only if} \quad \vdash_{\mathbf{LI}} \neg\neg A.$$

We have just proved that $\nvdash_{\mathbf{LI}} (\neg\neg A \Rightarrow A)$. Obviously $\models (\neg\neg A \Rightarrow A)$, so we know that $\neg\neg(\neg\neg A \Rightarrow A)$ must have a proof in **LI**.

We are going to prove that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg\neg A \Rightarrow A)$$

and that the structural rule $(contr \longrightarrow)$ is essential to the existence of its proof, i.e. that without it the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI**.

The following decomposition tree **T** is a proof of $\neg\neg(\neg\neg A \Rightarrow A)$ in **LI**.

T

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$first\ of\ 2\ choices:\ (\rightarrow \neg), (\rightarrow weak)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$first\ of\ 2\ choices: (contr \longrightarrow), (\neg \longrightarrow)$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$one\ of\ 2\ choices$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$one\ of\ 3\ choices$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$one\ of\ 2\ choices$$

$$\mid (\longrightarrow weak)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$one\ of\ 3\ choices$$

$$\mid (exch \longrightarrow)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$one\ of\ 3\ choices$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$one\ of\ 3\ choices$$

$$\mid (\longrightarrow \neg)$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$
$$one\ of\ 2\ choices$$
$$\mid (exch \longrightarrow)$$
$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$
$$one\ of\ 3\ choices$$
$$\mid (\neg \longrightarrow)$$
$$A \longrightarrow (\neg\neg A \Rightarrow A)$$
$$one\ of\ 3\ choices$$
$$\mid (\longrightarrow \Rightarrow)$$
$$\neg\neg A, A \longrightarrow A$$
$$axiom$$

Assume now that the rule ($contr \longrightarrow$) is not available. All possible decomposition trees are as follows.

T1

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow \neg)$$
$$one\ of\ 2\ choices$$
$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$only\ one\ choice$$
$$\longrightarrow (\neg\neg A \Rightarrow A)$$
$$\mid (\longrightarrow \Rightarrow)$$
$$one\ of\ 2\ choices$$
$$\neg\neg A \longrightarrow A$$
$$\mid (\longrightarrow weak)$$
$$only\ one\ choice$$
$$\neg\neg A \longrightarrow$$
$$\mid (\neg \longrightarrow)$$
$$only\ one\ choice$$
$$\longrightarrow \neg A$$
$$\mid (\longrightarrow \neg)$$
$$one\ of\ 2\ choices$$
$$A \longrightarrow$$
$$non-axiom$$

T2

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non-axiom$$

T3

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non-axiom$$

T4

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$]$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$only\ one\ choice$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non - axiom$$

This proves that the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI** without $(contr \longrightarrow)$ rule and hence this rule can't be eliminated.

sec:hproc

2.3 Proof Search Heuristic Method

Before we define a heuristic method of searching for proof in **LI** let's make some additional observations to the observations 1-5 from section 2.1.

Observation 6: Our goal while constructing the decomposition tree is to obtain axiom or indecomposable leaves. With respect to this goal the use logical decomposition rules has a priority over the use of the structural rules and we use this information while describing the proof search heuristic.

Observation 7: all logical decomposition rules $(\circ \rightarrow)$, where $\circ$ denotes any connective, must have a formula we want to decompose as the first formula at the decomposition node, i.e. if we want to decompose a formula $\circ A$, the node must have a form $\circ A, \Gamma \longrightarrow \Delta$. Sometimes it is necessary to decompose a formula within the sequence Γ first in order to find a proof.

For example, consider two nodes

$$n_1 = \neg\neg A, (A \cap B) \longrightarrow B$$

and

$$n_2 = (A \cap B), \neg\neg A \longrightarrow B.$$

We are going to see that the results of decomposing n_1 and n_2 differ dramatically.

Let's decompose the node n_1. Observe that the only way to be able to decompose the formula $\neg\neg A$ is to use the rule $(\rightarrow weak)$ first. The two possible decomposition trees that starts at the node n_1 are as follows.

$$\mathbf{T1}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\cap \rightarrow)$$

$$A, B \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

T2$_{n_1}$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, (A \cap B) \longrightarrow$$

$$\mid (\cap \rightarrow)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

Let's now decompose the node n_2. Observe that following our **Observation 6** we start by decomposing the formula $(A \cap B)$ by the use of the rule $(\cap \rightarrow)$ first. A decomposition tree that starts at the node n_2 is as follows.

T1$_{n_2}$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

This proves that the node n_2 is provable in **LI**, i.e.

$$\vdash_{\mathbf{LI}} \quad (A \cap B), \neg\neg A \longrightarrow B.$$

Of course, we have also that the node n_1 is also provable in **LI**, as one can obtain the node n_2 from it by the use of the rule $(exch \rightarrow)$.

Observation 8: the use of structural rules are important and necessary while we search for proofs. Nevertheless we have to use them on the "must" basis and set up some guidelines and priorities for their use.

For example, use of weakening rule discharges the weakening formula, and hence an information that may be essential to the proof. We should use it only when it is absolutely necessary for the next decomposition steps. Hence, the use of weakening rule $(\rightarrow weak)$ can, and should be restricted to the cases when it leads to possibility of the use of the negation rule $(\neg \rightarrow)$.

This was the case of the decomposition tree $\mathbf{T1}_{n_1}$. We used it as an necessary step, but still it discharged too much information and we didn't get a proof, when proof of the node existed.

In this case the first rule in our search should have been the exchange rule, followed by the conjunction rule (no information discharge) not the weakening (discharge of information) followed by negation rule. The full proof of the node n_1 is the following.

$$\mathbf{T3}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (exch \longrightarrow)$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

As a result of the **observations 1- 5** from section 2.1 and **observations 6 - 8** above we adopt the following.

Heuristic Procedure for Proof Search in LI.

For any $A \in \mathcal{F}$ we construct the set of decomposition trees $\mathbf{T}_{\rightarrow A}$ following the rules below.

Rules for Decomposition Tree Generation

1. Use first logical rules where applicable.

2. Use $(exch \rightarrow)$ rule to decompose, via logical rules, as many formulas on the left side of $\longrightarrow$ as possible.

3. Use $(\rightarrow weak)$ only on a "must" basis in connection with $(\neg \rightarrow)$ rule.

4. Use $(contr \rightarrow)$ rule as the last recourse and only to formulas that contain $\neg$ or $\Rightarrow$ as connectives.

5. Let's call a formula A to which we apply $(contr \rightarrow)$ rule **a contraction** formula.

6. The only contraction formulas are formulas containing $\neg$ between theirs logical connectives.

7. Within the process of construction of all possible trees use $(contr \rightarrow)$ rule only to contraction formulas.

8. Let C be a contraction formula appearing on the node n of the decomposition tree of $\mathbf{T}_{\rightarrow A}$. For any contraction formula C, any node n, we apply $(contr \rightarrow)$ rule the the formula C at most as many times as the number of sub-formulas of C.

If we find a tree with all axiom leaves we have a proof, i.e. $\vdash_{LI} A$ and if all (finite number) trees have a non-axiom leaf we have proved that proof of A does not exist, i.e. $\nvdash_{LI} A$.

3 Introduction to Modal S4 and S5 Logics

The non-classical logics can be divided in two groups: those that rival classical logic and those which extend it. The Lukasiewicz, Kleene, and Intuitionistic Logics are in the first group. The modal logics are in the second.

The rival logics do not differ from classical logic in terms of the language employed. Rather, rival logics differ in that certain theorems or tautologies of classical logic are rendered false, or not provable in them.

Perhaps the most notorious example of this is the law of excluded middle $(A \cup \neg A)$. This is provable in, and is a tautology of classical logic but is not provable in, and is not tautology of intuitionistic logic, or is not a tautology under any of the extensional logics semantics we have discussed.

Logics which extend classical logic sanction all the theorems of classical logic but, generally, supplement it in two ways. Firstly, the languages of these non-classical logics are extensions of those of classical logic, and secondly, the theorems of these non-classical logics supplement those of classical logic. Usually, such supplementation is provided by the enriched language. For example, modal logics are enriched by the addition of two new connectives that represent the meaning of *it is necessary that* and *it is possible that.* We use the notation **I** for *it is necessary that* and **C** for *it is possible that.* Other notations used are: ∇, N, L for *it is necessary that*, and $\diamondsuit$ P, M for *it is possible that.* The symbols N, L, P, M or alike, are often used in computer science investigations. The symbols ∇ and $\diamond$ were first to be used in modal logic literature, the symbols **I, C** come from algebraic and topological interpretation of modal logics. **I** corresponds to the *interior* of the set and **C** to its *closure.*

The idea of a modal logic was first formulated by an American philosopher, C.I. Lewis in 1918. He has proposed yet another interpretation of lasting consequences, of the logical implication. In an attempt to avoid, what some felt, the paradoxes of semantics for classical implication which accepts as true that a false sentence implies any sentence he created a notion of *a modal truth*, which lead to the notion of *modal logic.* The idea was to distinguish two sorts of truth: *necessary* truth and mere *possible (contingent)* truth. A *possibly* true sentence is one which, though true, could be false. A necessary truth is hence the one which could not be otherwise; a contingent (possible) truth is one which could. The distinction between them is a metaphysical one and should not be confused with the distinction between *a priori* and *a posteriori* truths. An *a priori* truth is one which can be *known* independently of experience, and an *a posteriori* truth is one which cannot. Such notions appeal to epistemic considerations and the whole area of modal logics bristles with philosophical difficulties and hence the numbers of logics have been created. Unlike the classical connectives, the modal connectives do not admit of truth-functional interpretation. This was the reason for which modal logics was first developed as a proof systems, with intuitive notion of semantics expressed by the set of adopted axioms.

The first semantics, and hence the proofs of the completeness theorems came some 20 years later. It took yet another 25 years for discovery and development of the second more general approach to the semantics. These are two established ways of interpret modal connectives, i.e. to define modal semantics.

The historically first one is due to Mc Kinsey and Tarski (1944, 1946). It is a topological interpretation that provides a powerful mathematical interpretation of some of them, namely S4 and S5. It connects the modal notion of *necessity* with the topological notion of *interior* of a set, and the notion of *possibility* with the notion of its *closure* . Our choice of symbols **I** and **C** for modal connectives comes from this interpretation. The topological interpretation powerful as it is, is less universal in providing models for other modal logics. The most recent one is due to Kripke (1964). It uses the notion *possible world.* Roughly, we say that $\mathbf{C}A$ is true if A is true in some possible world, called actual world, and $\mathbf{I}A$ is true if A is true in every possible world.

We present the formal definition later in this chapter, but this intuitive meaning can be useful in unconvincing ourselves about validity (or sense) of adopted axioms and rules of inference.

As we have already mentioned, modal logics were first developed, as was the intuitionistic logic, in a form of proof systems only. First Hilbert style modal proof system was published by Lewis and Langford in 1932. They presented a formalization for two modal logics, which they called S1 and S2. They also outlined three other proof systems, called S3, S4, and S5.

In 1933 Gödel worked with Heyting's "sentential logic" proof system, what we are calling now Intuitionistic logic. He considered a particular modal proof system and asserted that theorems of Heyting's "sentential logic" could be obtained from it by using a certain translation. His presentation of the discovered proof system, now known as S4 logic, was particularly elegant.

Since then hundreds of modal logics have been created. There are some standard texts in the subject. These are, between the others: Hughes and Cresswell (1969) for philosophical motivation for various modal logics and Intuitionistic logic, Bowen (1979) for a detailed and uniform study of Kripke models for modal logics, Segeberg (1971) for excellent classification, and Fitting (1983), for extended and uniform studies of automated proof methods for classes of modal logics.

Hilbert Style Modal Proof Systems

We present here Hilbert style formalization for S4 and S5 logics due to Mc Kinsey and Tarski (1948) and Rasiowa and Sikorski (1964). We also discuss the relationship between S4 and S5, and between the Intuitionistic logic and S4 modal logic, as first observed by Gödel.

They stress the connection between S4, S5 and topological spaces which constitute models for them. Hence the use of symbols **I**, **C** for necessity and possibility, respectively. The connective **I** corresponds to the symbol denoting a topological interior of a set and **C** to the closure of a set.

Modal Language

We add to the propositional language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ two extra one argument connectives **I** and **C**. I.e. we adopt

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}} \tag{39}$$ mlang

as our modal language. We read a formula $\mathbf{I}A$, $\mathbf{C}A$ as *necessary* A and *possible* A, respectively.

The language is common to all modal logics. Modal logics differ on a choice of axioms and rules of inference, when studied as proof systems and on a choice of semantics.

McKinsey, Tarski (1948)

As modal logics extend the classical logic, any modal logic contains two groups of axioms: classical and modal.

Axioms Group 1: classical axioms

Any modal logic adopts as its classical axioms any complete set of axioms for a classical propositional logic.

Axioms Group 2: modal axioms

M1 $(\mathbf{I}A \Rightarrow A)$,

M2 $(\mathbf{I}(A \Rightarrow B) \Rightarrow (\mathbf{I}A \Rightarrow \mathbf{I}B))$,

M3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$,

M4 $(\mathbf{C}A \Rightarrow \mathbf{IC}A)$.

Rules of inference

We adopt the Modus Ponens (MP)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

and an additional modal rule (I) introduced by Gödel

$$(I)\ \frac{A}{\mathbf{I}A}$$

referred to as *necessitation.*

We define modal proof systems S4 and S5 as follows.

$$S4 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ M1 - M3,\ (MP),\ (I)\ ), \tag{40}$$ s4

$$S5 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ M1 - M4,\ (MP),\ (I)\ ). \tag{41}$$ s5

Observe that the axioms of S5 extend the axioms of S4 and both system share the same inference rules, hence we have immediately the following.

mfact **Fact 3**

For any formula $A \in \mathcal{F}$, *if* $\vdash_{S4} A$, *then* $\vdash_{S5} A$.

Rasiowa, Sikorski (1964)

It is often the case, and it is in our systems, that modal connectives are expressible by each other, i.e. that we can adopt one of them and define the other as follows.

$$\mathbf{I}A = \neg\mathbf{C}\neg A, \tag{42}$$ mI

and

$$\mathbf{C}A = \neg\mathbf{I}\neg A. \tag{43}$$ mC

The equality sign in equations (42), and (43) means that we replace the formula in left side of the equation by the formula in the right side anywhere where the left side (or right side) formula is appears as a sub formula of a formula of $\mathcal{L}$. In modal logics S4 and S5 the connective **C** is expressible by $\neg$ and **I**, as stated above by (43), we hence assume now that the language $\mathcal{L}$ contains only one modal connective **I**.

Language

$$\mathcal{L} = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \mathbf{I}\}}. \tag{44}$$ l1

There are, as before, two groups of axioms: classical and modal.

Axioms Group 1: classical axioms

We adopt as classical axioms any complete set of axioms for a classical propositional logic.

Axioms Group2: modal axioms

R1 $((\mathbf{I}A \cap \mathbf{I}B) \Rightarrow \mathbf{I}(A \cap B))$,

R2 $(\mathbf{I}A \Rightarrow A)$,

R3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$,

R4 $\mathbf{I}(A \cup \neg A)$,

R5 $(\neg \mathbf{I} \neg A \Rightarrow \mathbf{I} \neg \mathbf{I} \neg A)$

Rules of inference

We adopt the Modus Ponens (MP)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

and an additional modal rule (RI)

$$(RI)\ \frac{(A \Rightarrow B)}{(\mathbf{I}A \Rightarrow \mathbf{I}B)}.$$

We define modal logic proof systems $RS4$, $RS5$ as follows.

$$RS4 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ R1 - R4,\ (MP),\ (RI)\ ). \tag{45}$$ rs4

$$RS5 = (\ \mathcal{L},\ \mathcal{F},\ \text{classical axioms},\ R1 - R5,\ (MP),\ (RI)\ ). \tag{46}$$ rs5

Observe that the axioms of RS5 extend, as the axioms of TS5, the axioms of TS4 and both system share the same inference rules, hence we have immediately the following.

rfact **Fact 4**

For any formula $A \in \mathcal{F}$, *if* $\vdash_{RS4} A$, *then* $\vdash_{RS5} A$.

3.1 Algebraic Semantics for S4 and S5

The McKinsey, Tarski proof systems (40), (41), and Rasiowa, Sikorski proof systems (45), (45) for modal logics S4, S5 are complete with the respect to both algebraic topological semantics, and Kripke semantics.

We shortly discuss the topological semantics, and algebraic completeness theorems and leave the Kripke semantics for the reader to explore from other, multiple sources.

The topological semantics was initiated by McKinsey, Tarski's (1946, 1948) and consequently developed and examined by many authors into a field of *Algebraic Logic.*They are presented in detail in now classic algebraic logic books: *"Mathematics of Metamathematics"*, Rasiowa, Sikorski (1964) and *"An Algebraic Approach to Non-Classical Logics"*, Rasiowa (1974).

We want to point out that the first idea of a connection between modal propositional calculus and topology is due to Tang Tsao -Chen, (1938) and Dugunji (1940).

Here are some basic definitions.

Boolean Algebra
An abstract algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg) \tag{47}$$

Balg

is said to be a *Boolean algebra* if it is a distributive lattice and every element $a \in B$ has a complement $\neg a \in B$.

Topological Boolean algebra
By a *topological Boolean algebra* we mean an abstract algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg,\ I), \tag{48}$$

Talg

where $(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$ is a Boolean algebra (47) and, moreover, the following conditions hold for any $a, b \in B$.

$$I(a \cap b) = Ia \cap Ib, \quad Ia \cap a = Ia, \quad IIa = Ia, \ \text{ and } \ I1 = 1. \tag{49}$$

Icond

The element Ia is called a **interior** of a. The element $\neg I \neg a$ is called a **closure of a** and will be denoted by Ca. Thus the operations I and C are such that

$$Ca = \neg I \neg a \ \text{ and } \ Ia = \neg C \neg a.$$

In this case we write the topological Boolean algebra (48) as

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg,\ I,\ C). \tag{50}$$

Talg1

It is easy to prove that in in any topological Boolean algebra (50) the following conditions hold for any $a, b \in B$.

$$C(a \cup b) = Ca \cup Cb, \quad Ca \cup a = Ca, \quad CCa = Ca, \quad \text{and} \quad C0 = 0. \tag{51}$$ Ccond

If X is a topological space with an interior operation I, then the family $\mathcal{P}(X)$ of all subsets of X is a topological Boolean algebra with $1 = X$, the operation $\Rightarrow$ defined by the formula

$$Y \Rightarrow Z = (X - Y) \cup Z, \quad \text{for all subsets } Y, Z \text{of } X,$$

set-theoretical operations of union, intersection, complementation, and the interior operation I. Obviously, every sub algebra of this algebra is a topological Boolean algebra, called a *topological field of sets* or, more precisely, a *topological field* of subsets of X.

Given a topological Boolean algebra (47) $(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$. The element $a \in B$ is said to be **open** (**closed**) if $a = Ia$ ($a = Ca$).

Clopen Topological Boolean algebra
A topological Boolean algebra (50)

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C\ ).$$

such that every **open** element is **closed** and every **closed** element is **open**, i.e. such that for any $a \in B$

$$CIa = Ia \quad \text{and} \quad ICa = Ca \tag{52}$$ clopen

is called a *clopen topological Boolean algebra.*

We loosely say that a formula A of a modal language is a modal $S4$ **tautology** if and only if any topological Boolean algebra (50) is a *model* for A.
We say that A is a modal $S5$ **tautology** if and only if any **clopen** topological Boolean algebra (52) is a *model* for A.

We put it formally as follows.

Amodel **Definition 4 (Modal Algebraic Model)**

For any formula A of a modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ and for any topological Boolean algebra $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg,\ I,\ C)$,

the algebra $\mathcal{B}$ is a **model** *for the formula A and denote it by*

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$.

Mtaut **Definition 5 ($S4, S5$ Tautology)**

The formula A is a modal $S4$ **tautology** *($S5$* **tautology***) and is denoted by*

$$\models_{S4} A \quad (\models_{S5} A)$$

if and only if for all **topological** *Boolean algebras (***clopen topological** *Boolean algebras) $\mathcal{B}$ we have that*

$$\mathcal{B} \models A \quad (\mathcal{B} \models A).$$

In Algebraic Logic the notion of tautology is often defined using a notion "a formula A is **valid** in an algebra $\mathcal{B}$ ".It is formally defined in our case as as follows.

valid **Definition 6**

A formula A **is valid** *in a topological Boolean algebra $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C)$, if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$.*

Directly form definitions 5, 6 we get the following.

Fact 5 *For any formula A, $\models_{S4} A$ $(\models_{S5} A)$ if and only if A* **is valid** *in all topological Boolean algebras (A* **is valid** *in all clopen topological Boolean algebras).*

We write $\vdash_{S4} A$ and $\vdash_{S5} A$ do denote any proof system for modal S4, S5 logics and in particular the proof systems (40), (45), and (41), (46), respectively.

cs4 **Theorem 17 (Completeness Theorem)** *For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$,*

$$\vdash_{S4} A \ \textit{ if and only if } \ \models_{S4} A,$$

$$\vdash_{S5} A \ \textit{ if and only if } \ \models_{S5} A.$$

The completeness for $S4$ follows directly from the Theorem 18. The completeness for $S5$ follows from the $S4$ completeness and Embedding Theorems 22, 23. It also can be easily proved independently by adopting the Algebraic Completeness Theorem proof for $S4$ to clopen topological algebras.

cthm **Theorem 18 (Algebraic Completeness Theorem)**

For any formula A of the modal language $\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ the following conditions are equivalent.

(i) $\vdash_{S4} A$,

(ii) $\models_{S4} A$,

(iii) *A is valid in every topological field of sets $\mathcal{B}(X)$,*

(iv) *A is valid in every topological Boolean algebra* $\mathcal{B}$ *with at most* 2^{2^r} *elements, where r is the number of all sub formulas of A,*

(iv) $v^*(A) = X$ *for every variable assignment v in the topological field of sets* $\mathcal{B}(X)$ *of all subsets of a dense-in -itself metric space* $X \neq \emptyset$ *(in particular of an n-dimensional Euclidean space X).*

On S4 derivable disjunction

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when neither A nor B is a tautology. The tautology $(A \cup \neg A)$ is the simplest example. This does not hold for the intuitionistic logic. We have a similar theorem for modal S4 logic, as proved by McKinsey and Tarski.

mdis **Theorem 19**

A disjunction $(IA \cup IB)$ *is S4 provable if and only if either A or B is S4 provable, i.e.*

$$\vdash_{S4} (IA \cup IB) \quad \textit{if and only if} \quad \vdash_{S4} A \quad \textit{or} \ \vdash_{S4} B.$$

The proof follows directly from the Completeness Theorem 18 and the following semantical, proof system independent version of the theorem 19.

Theorem 20 (McKinsey, Tarski, 1948)

For any $A \in \mathcal{F}$,

$$\models_{S4} (IA \cup IB) \ \textit{ if and only if } \ \models_{S4} A \textit{ or } \models_{S4} B.$$

The completeness theorem allows us to formulate theorems about logics in terms independent of the proof system considered. In this sense the notion of tautology is more general then the notion of provability. This is why often we use rather the tautology formulation of the known facts about the logic and their relationships instead of the notion of provability.

Following the Completeness Theorem 18 we get a semantical version of the theorem 3.

tfact **Theorem 21**

For any formula $A \in \mathcal{F}$,

$$\textit{if} \quad \models_{S4} A, \quad \textit{then} \quad \models_{S5} A.$$

Consider a modal language $\mathcal{L}$ with both modal connectives, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, \mathbf{I}, \mathbf{C}\}}.$$

The above theorem21 says that the S4 tautologies form a subset of S5 tautologies. We have even a more powerful relationship, namely the following.

eb1 **Theorem 22 (Embedding 1)**

For any formula $A \in \mathcal{F}$,

$$\models_{S4} A \quad \textit{if and only if} \quad \models_{S5} \mathbf{IC}A,$$

$$\vdash_{S4} A \quad \textit{if and only if} \quad \vdash_{S5} \mathbf{IC}A.$$

eb2 **Theorem 23 (Embedding 2)**

For any formula $A \in \mathcal{F}$,

$$\models_{S5} A \quad \textit{if and only if} \quad \models_{S4} \mathbf{ICI}A.$$

$$\vdash_{S5} A \quad \textit{if and only if} \quad \vdash_{S4} \mathbf{ICI}A.$$

eb3 **Theorem 24 (Embedding 3)**

For any formula $A \in \mathcal{F}$,

$$\textit{if} \ \models_{S5} A, \quad \textit{then} \quad \models_{S4} \neg\mathbf{I}\neg A.$$

The fist proof of the above embedding theorems was given by Matsumoto in 1955. Provability. Fitting semantical 1983 Ohnishi and Matsumoto 1957/59 Gentzen Methods in Modal Calculi Osaka Mathematical Journal 9.113 -130

3.2 S4 and Intuitionistic Logic, S5 and Classical Logic

As we have said in the introduction, Gödel was the first to consider the connection between the intuitionistic logic and a logic which was named later S4. His proof was purely syntactic in its nature, as semantics for neither intuitionistic logic nor modal logic S4 had not been invented yet.

The algebraic proof of this fact, was first published by McKinsey and Tarski in 1948. We now define the mapping establishing the connection (definition 7) and refer the reader to Rasiowa and Sikorski book "Mathematics of Metamathematics" for its proof.

Let $\mathcal{L}$ be a propositional language of modal logic, as defined by (44), i.e the language

$$\mathcal{L} = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \mathbf{I}\}}.$$

Let $\mathcal{L}_0$ be a language obtained from $\mathcal{L}$ by elimination of the connective $\mathbf{I}$ and by the replacement the negation connective $\neg$ by the intuitionistic negation, which we will denote here by a symbol $\sim$. Such obtained language

$$\mathcal{L}_0 = \mathcal{L}_{\{\cap, \cup, \Rightarrow, \sim\}} \tag{53}$$ 12

is a propositional language of the intuitionistic logic.

In order to establish the connection between the languages (44) and (53), and hence between modal and intuitionistic logic, we define a mapping f which to every formula $A \in \mathcal{F}_0$ of $\mathcal{L}_0$ assigns a formula $f(A) \in \mathcal{F}$ of $\mathcal{L}$.

We define a mapping f as follows.

mapping **Definition 7 (Gódel - Tarski)**

A function $f : \mathcal{F}_0 \quad \to \mathcal{F}$ be such that

$$f(a) = \mathbf{I}a \quad for \quad any \quad a \in VAR,$$

$$f((A \Rightarrow B)) = \mathbf{I}(f(A) \Rightarrow f(B)),$$
$$f((A \cup B)) = (f(A) \cup f(B)),$$
$$f((A \cap B)) = (f(A) \cap f(B)),$$
$$f(\sim A) = \mathbf{I}\neg f(A),$$

where A, B denote any formulas in $\mathcal{L}_0$ is called a **Gödel - Tarski mapping**.

Example

Let A be a formula

$$((\sim A \cap \sim B) \Rightarrow \sim (A \cup B))$$

and f be the mapping of definition 7. We evaluate $f(A)$ as follows

$$f((\sim A \cap \sim B) \Rightarrow \sim (A \cup B)) =$$

$$\mathbf{I}(f(\sim A \cap \sim B) \Rightarrow f(\sim (A \cup B)) =$$
$$\mathbf{I}((f(\sim A) \cap f(\sim B)) \Rightarrow f(\sim (A \cup B)) =$$
$$\mathbf{I}((\mathbf{I}\neg fA \cap \mathbf{I}\neg fB) \Rightarrow \mathbf{I}\neg f(A \cup B)) =$$
$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(fA \cup fB)) =$$
$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(A \cup B)).$$

We use notation $\vdash_I A$ do denote the fact that A is intuitionistically provable, i.e. provable in any intuitionistic proof system I.

With these hypotheses and notation the following theorem holds.

pmint **Theorem 25**

Let f be the Gödel mapping (definition 7). For any formula A of $\mathcal{L}_0$,

$$\vdash_I A \quad \textit{if and only if} \quad \vdash_{S4} f(A),$$

where I, $S4$ denote any proof systems for intuitionistic and and S4 logic, respectively.

In order to establish the connection between the modal logic and classical logic we consider the Gódel - Tarski mapping (definition **??**) between the modal language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$ and its classical sub-language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Now with every classical formula A we associate a modal formula $f(A)$ defined by induction on the length of A as follows:

$$f(a) = \mathbf{I}a, \quad f((A \Rightarrow B)) = \mathbf{I}(f(A) \Rightarrow f(B)),$$

$$f((A \cup B)) = (f(A) \cup f(B)),\ f((A \cap B)) = (f(A) \cap f(B)),\ f(\neg A) = \mathbf{I}\neg f(A).$$

We use notation $\vdash_H \ A$ do denote the fact that A is classically provable, i.e. provable in any proof system for classical propositional logic.

The following theorem established relationship between classical logic and modal $S5$.

Theorem 26

Let f be the Gödel mapping (definition 7) between $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ and $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$. For any formula A of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$,

$$\vdash_H \ A \quad \textit{if and only if} \quad \vdash_{S5} f(A),$$

where H, $S5$ denote any proof systems for classical and and S5 modal logic, respectively.

4 Homework Problems

1. The algebraic models for the intuitionistic logic are defined in terms of *Pseudo-Boolean Algebras* in the following way. A formula A is said to be an intuitionistic tautology if and only if $v \models A$, for all v and all Pseudo-Boolean Algebras, where v maps VAR into universe of a Pseudo-Boolean Algebra. I.e. A is an intuitionistic tautology if and only if it is true in all Pseudo-Boolean Algebras under all possible variable assignments.

   A 3 element Heyting algebra as defined in chapter **??** is a 3 element Pseudo-Boolean Algebra.

   (i) Show that the 3 element Heyting algebra is a model for all formulas (12) - (27).

   (ii) Determine for which of the formulas (28) - (34) the 3 element Heyting algebra acts as a counter-model.

2. Find proofs by constructing proper decomposition trees in Gentzen System **LI** of axioms **A1 - A11** of the proof system I defined by (4).

3. The completeness with respect to algebraic semantics of system **LI** follows from the Hauptzatz Theorem and the completeness theorem 2. The proof is a straightforward adaptation of the proof of the completeness theorem for **LK** included in chapter **??**. Write carefully all steps of the proof of completeness theorem for **LI** .

4. Find proofs by constructing proper decomposition trees in Gentzen System **LI** of the intuitionistic tautologies (12) - (27).

5. Show that none of the formulas (28) - (34) is provable in **LI**.

6. Find proofs by constructing proper decomposition trees in Gentzen System **LI** of double negation of all of the formulas (28) - (34).

7. Give the proof of the Glivenko theorem 7, i.e. prove that any formula A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable.

8. Give few examples of formulas illustrating that the following theorems hold.

   Gödel (1) For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is intuitionistically provable.

9. Give examples of formulas illustrating that the following theorems hold.

   Gödel (2) If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable.

10. Use the Completeness Theorem 18 to show that the following proof system **C**S4 is a complete proof system for the modal logic S4.

    We adopt the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$. We adopt, as before, two groups of axioms: classical and modal.
    Group 1: we take any complete set of axioms for a classical propositional logic. Group 2: the following modal axioms.
    C1 $(\mathbf{C}(A \cup B) \Rightarrow (\mathbf{C}A \cup \mathbf{C}B))$,

    C2 $(A \Rightarrow \mathbf{C}A)$,

    C3 $(\mathbf{C}\mathbf{C}A \Rightarrow \mathbf{C}A)$,

    C4 $\mathbf{C}(A \cap \neg A)$.

    Rules of inference: we adopt the Modus Ponens (MP) and an additional rule,

    $$(\mathbf{C})\ \frac{(A \Rightarrow B)}{(\mathbf{C}\neg B \Rightarrow \mathbf{C}\neg A)}.$$

    We define the proof system $CS4$ as follows

    $$CS4 = (\ \mathcal{L}, \mathcal{F},\ \text{classical}\ \ \text{axioms},\ C1 - C4,\ (MP),\ (\mathbf{C})\ ).$$

11. Evaluate $f(A)$, where f is the Gödel- Tarski mapping (definition 7), for all the formulas A listed below.

 (i) $(\neg A \cup \neg B) \Rightarrow (\neg A \cap \neg B))$

 (ii) $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

 (iii) $((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$

 (iv) $(\neg\neg\neg A \Rightarrow \neg A)$

 (v) $(\neg A \Rightarrow \neg\neg\neg A)$

 (vi) $(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$

 (vii) $((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$

12. Use the Completeness Theorem 18 and Embedding Theorems 22, 23 to show the following.

 (i) For any formula A, $\vdash_{RS4}$ A, if and only if $vdash_{RS5}$ $\mathbf{I}\neg\mathbf{I}\neg A$, where RS4, RS5 are proof system (45) and (46).

 (ii) For any formula $A, \vdash_{S5}$ A, if and only if $\models_{S4}$ $\mathbf{I}\neg\mathbf{I}\neg\mathbf{I}A$, where S4, S5 are proof system (40) and (41).