Chapter 6: Definability of Connectives, Equivalence of Languages

Definition of Logical equivalence:

For any formulas $A, B$,

$$A \equiv B \iff \models (A \iff B).$$

Property:

$$A \equiv B \iff \models (A \Rightarrow B) \text{ and } \models (B \Rightarrow A).$$
**Substitution Theorem** Let $B_1$ be obtained from $A_1$ by substitution of a formula $B$ for one or more occurrences of a sub-formula $A$ of $A_1$.

**We denote** it as

$$B_1 = A_1(A/B).$$

**Then** the following holds.

*If* $A \equiv B$, *then* $A_1 \equiv B_1$, 
The next set of equivalences, or corresponding tautologies, deals with what is called a definability of connectives in classical semantics.

For example, a tautology

$$
\models ((A \Rightarrow B) \iff (\neg A \cup B))
$$

makes it possible to define implication in terms of disjunction and negation.

We state it in a form of logical equivalence as follows.

Definability of Implication in terms of negation and disjunction:

$$(A \Rightarrow B) \equiv (\neg A \cup B)$$
We use logical equivalence notion, instead of the tautology notion, as it makes the manipulation of formulas much easier.

**Definability of Implication** equivalence allows us, by the force of **Substitution Theorem to replace** any formula of the form \((A \Rightarrow B)\) placed anywhere in another formula by a formula \((\neg A \cup B)\).

**Hence we transform** a given formula containing implication into an logically equivalent formula that does contain implication (but contains negation and disjunction).
Example 1  We transform (via Substitution Theorem) a formula

\[((C \Rightarrow \neg B) \Rightarrow (B \cup C))\]

into its logically equivalent form not containing \(\Rightarrow\) as follows.

\[((C \Rightarrow \neg B) \Rightarrow (B \cup C))\]

\[\equiv (\neg (C \Rightarrow \neg B) \cup (B \cup C))\]

\[\equiv (\neg (\neg C \cup B) \cup (B \cup C)).\]

We get

\[((C \Rightarrow \neg B) \Rightarrow (B \cup C))\]

\[\equiv (\neg (\neg C \cup B) \cup (B \cup C)).\]
It means that we can, by the Substitution Theorem transform a language

\[ \mathcal{L}_1 = \mathcal{L}\{\neg, \cap, \Rightarrow\} \]

into a language

\[ \mathcal{L}_2 = \mathcal{L}\{\neg, \cap, \cup\} \]

with all its formulas being logically equivalent.

We write it as the following condition.

C1: for any formula \( A \) of \( \mathcal{L}_1 \), there is a formula \( B \) of \( \mathcal{L}_2 \), such that \( A \equiv B \).
Example 2: Let $A$ be a formula

\[(\neg A \cup (\neg A \cup \neg B))\]

We use the definability of implication equivalence to eliminate disjunction as follows

\[(\neg A \cup (\neg A \cup \neg B)) \equiv (\neg A \cup (A \Rightarrow \neg B))\]

\[\equiv (A \Rightarrow (A \Rightarrow \neg B)).\]

Observe, that we can’t always use the equivalence $(A \Rightarrow B) \equiv (\neg A \cup B)$ to eliminate any disjunction.

For example, we can’t use it for a formula

\[A \equiv ((a \cup b) \cap \neg a).\]
In order to be able to transform any formula of a language containing disjunction (and some other connectives) into a language with negation and implication (and some other connectives), but without disjunction we need the following logical equivalence.

**Definability of Disjunction** in terms of negation and implication:

\[(A \cup B) \equiv (\neg A \Rightarrow B)\]
Example 3 Consider a formula $A$

$$(a \cup b) \cap \neg a).$$

We transform $A$ into its logically equivalent form not containing $\cup$ as follows.

$$(a \cup b) \cap \neg a) \equiv ((\neg a \Rightarrow b) \cap \neg a).$$

In general, we transform the language $\mathcal{L}_2 = \mathcal{L}\{\neg, \cap, \cup\}$ to the language $\mathcal{L}_1 = \mathcal{L}\{\neg, \cap, \Rightarrow\}$ with all its formulas being logically equivalent.
We write it as the following condition. 

**C1:** for any formula $C$ of $\mathcal{L}_2$, there is a formula $D$ of $\mathcal{L}_1$, such that $C \equiv D$.

The languages $\mathcal{L}_1$ and $\mathcal{L}_2$ for which we the conditions **C1, C2** hold are called **logically equivalent**.

We denote it by 

$$\mathcal{L}_1 \equiv \mathcal{L}_2.$$ 

A general, formal definition goes as follows.
**Definition of Equivalence of Languages**

Given two languages: \( \mathcal{L}_1 = \mathcal{L}_{CON_1} \) and \( \mathcal{L}_2 = \mathcal{L}_{CON_2} \), for \( CON_1 \neq CON_2 \).

**We say** that they are logically equivalent, i.e.

\[
\mathcal{L}_1 \equiv \mathcal{L}_2
\]

if and only if the following conditions **C1**, **C2** hold.

**C1:** For every formula \( A \) of \( \mathcal{L}_1 \), there is a formula \( B \) of \( \mathcal{L}_2 \), such that

\[
A \equiv B,
\]

**C2:** For every formula \( C \) of \( \mathcal{L}_2 \), there is a formula \( D \) of \( \mathcal{L}_1 \), such that

\[
C \equiv D.
\]
Example 4  To prove the logical equivalence of the languages

\[ \mathcal{L}\{\neg, \cup\} \equiv \mathcal{L}\{\neg, \Rightarrow\} \]

we need two definability equivalences:

**implication** in terms of disjunction and negation,

**disjunction** in terms of implication and negation, and the **Substitution Theorem**.
Example 5 To prove the logical equivalence of the languages

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cap, \cup\}}$$

we need only the definability of implication equivalence.

It proves, by Substitution Theorem that for any formula \(A\) of

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

there is \(B\) of \(\mathcal{L}_{\{\neg, \cap, \cup\}}\) that equivalent to \(A\), i.e.

$$A \equiv B$$

and condition \(C1\) holds.

Observe, that any formula \(A\) of language

$$\mathcal{L}_{\{\neg, \cap, \cup\}}$$
is also a formula of

\[ \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \]

and of course

\[ A \equiv A, \]

so \textbf{C2} also holds.
The logical equalities below

**Definability of Conjunction** in terms of implication and negation

\[(A \cap B) \equiv \neg(A \Rightarrow \neg B),\]

**Definability of Implication** in terms of conjunction and negation

\[(A \Rightarrow B) \equiv \neg(A \cap \neg B),\]

and the **Substitution Theorem** prove that

\[\mathcal{L}\{\neg, \cap\} \equiv \mathcal{L}\{\neg, \Rightarrow\}.\]
Exercise 1

(a) Prove that
\[ \mathcal{L}_{\{\cap, \neg}\} \equiv \mathcal{L}_{\{\cup, \neg}\}. \]

(b) Transform a formula \( A = \neg(\neg(a \cap \neg b) \cap a) \)
    of \( \mathcal{L}_{\{\cap, \neg}\} \) into a logically equivalent formula
    \( B \) of \( \mathcal{L}_{\{\cup, \neg}\} \).

(c) Transform a formula
    \[ A = (((a \cup \neg b) \cup a) \cup (a \cup \neg c)) \]
    of \( \mathcal{L}_{\{\cup, \neg}\} \) into
    a formula \( B \) of \( \mathcal{L}_{\{\cap, \neg}\} \), such that \( A \equiv B \).

(d) Prove/disprove: \( \models \neg(\neg(a \cap \neg b) \cap a) \).

(e) Prove/disprove:
    \( \models (((a \cup \neg b) \cup a) \cup (a \cup \neg c)). \)
Solution (a) True due to the Substitution Theorem and two definability of connectives equivalences:

\[(A \cap B) \equiv \neg(\neg A \cup \neg B), \quad (A \cup B) \equiv \neg(\neg A \cap \neg B).\]

Solution (b)

\[\neg(\neg(\neg a \cap \neg b) \cap a)\]

\[\equiv \neg(\neg \neg(\neg a \cup \neg b) \cap a)\]

\[\equiv \neg((a \cup b) \cap a)\]

\[\equiv \neg((a \cup b) \cup \neg a).\]

The formula \(B\) of \(L_{\{\cup, \neg}\}\) equivalent to \(A\) is

\[B \equiv \neg((a \cup b) \cup \neg a).\]
Solution (c)

\[ (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c)) \]

\[ \equiv (((\neg(\neg \neg a \cap \neg \neg b) \cup a) \cup \neg(\neg a \cap \neg \neg c)) \]

\[ \equiv ((\neg(a \cap b) \cup a) \cup \neg(\neg a \cap c)) \]

\[ \equiv (\neg(\neg(\neg a \cap b) \cap \neg a) \cup \neg(\neg a \cap c)) \]

\[ \equiv (\neg((a \cap b) \cap \neg a) \cup \neg(\neg a \cap c)) \]

\[ \equiv \neg(\neg(\neg((a \cap b) \cap \neg a) \cap \neg \neg(\neg a \cap c))) \]

\[ \equiv \neg(((a \cap b) \cap \neg a) \cap (\neg a \cap c)) \]
There are two formulas $B$ of $\mathcal{L}_{\{\cap, \neg\}}$, such that $A \equiv B$.

$B = B_1 = \neg(\neg
\neg((a \cap b) \cap \neg a) \cap \neg \neg(\neg a \cap c)),$

$B = B_2 = \neg((a \cap b) \cap \neg a) \cap (\neg a \cap c)).$
Solution (d)

\[ \not \equiv \neg (\neg a \land \neg b) \land a \]

Our formula \( A \) is logically equivalent, as proved in (c) with the formula
\[ B = \neg (\neg (a \cup b) \cup \neg a). \]

Consider any truth assignment \( v \), such that \[ v(a) = F \] , then
\[ (\neg (a \cup b) \cup T) = T, \]
and hence \( v^*(B) = F. \)
Solution (e)

\[ \models (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c)) \]

because it was proved in (c) that

\[ (((\neg a \cup \neg b) \cup a) \cup (a \cup \neg c)) \equiv \neg (((a \cap b) \cap \neg a) \cap (\neg a \cap c)) \]

and obviously the formula

\[ (((a \cap b) \cap \neg a) \cap (\neg a \cap c)) \]

is a contradiction.

Hence its negation is a tautology.
Exercise 2 Prove by transformation, using proper logical equivalences that

1. 
   \[ \neg (A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)), \]

2. 
   \[ ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \]
   \[ \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)). \]
Solution 1.

\[ \neg (A \iff B) \]

\[ \equiv^{def} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \]

\[ \equiv^{de \ Morgan} (\neg (A \Rightarrow B) \cup \neg (B \Rightarrow A)) \]

\[ \equiv^{neg \ impl} ((A \cap \neg B) \cup (B \cap \neg A)) \]

\[ \equiv^{commut} ((A \cap \neg B) \cup (\neg A \cap B)) \].

Solution 2.

\[ ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \]

\[ \equiv^{impl} (\neg (B \cap \neg C) \cup (\neg A \cup B)) \]

\[ \equiv^{de \ Morgan} ((\neg B \cup \neg \neg C) \cup (\neg A \cup B)) \]

\[ \equiv^{neg} ((\neg B \cup C) \cup (\neg A \cup B)) \]

\[ \equiv^{impl} ((B \Rightarrow C) \cup (A \Rightarrow B)). \]