PART 1: DEFINITIONS

D1 Given a language $\mathcal{L}_{\{\Rightarrow, \cup, \cap, \neg\}}$ and a formula $A$ of this language.

$\models A$ if and only if $v \models A$ for all truth assignment $v : \text{VAR} \rightarrow \{T, F\}$

D2 Given formula $A \in \mathcal{F}$ of $\mathcal{L}_{\{\Rightarrow, \cup, \cap, \neg\}}$.

Write definition of $v$ is a restricted model for $A$.

A restricted MODEL for the formula $A$ is any function $w : \text{VAR}_A \rightarrow \{T, F\}$ such that $w(A) = T$, where $\text{VAR}_A$ is the set of all propositional variables appearing in $A$.

D3 Given a proof system $S = (\mathcal{L}, \mathcal{E}, \text{LA}, R)$ and an expression $E \in \mathcal{E}$.

$\vdash_S E$ if and only if there is a sequence $E_1, E_2, \ldots, E_n$ of expressions from $\mathcal{E}$, such that $n \geq 1$, and for each $1 < i \leq n$, either $E_i \in \text{LA}$ or $E_i$ is a direct consequence of some of the preceding expressions in $E_1, E_2, \ldots, E_n$ by virtue of one of the rules of inference $r \in R$.

D4 A proof system $S = (\mathcal{L}, \mathcal{E}, \text{LA}, R)$ is complete under a semantics $\mathcal{M}$ if and only if the following holds for any expression $E \in \mathcal{E}$.

$\vdash_S E$ if and only if $\models_{\mathcal{M}} E$.

D5 Write definition: A non-empty set $G \subseteq \mathcal{F}$ consistent under classical semantics.

A non-empty set $G \subseteq \mathcal{F}$ of formulas is called consistent if and only if $G$ has a model.

We can also say:

$G \subseteq \mathcal{F}$ is consistent if and only if there is a truth assignment $v$ such that $v \models G$,

or we say:

$G \subseteq \mathcal{F}$ is consistent if and only if there is $v$ such that $v^*(A) = T$ for all $A \in G$.

PART 2: PROBLEMS

PROBLEM 1

Given a set of formulas $G = \{(a \Rightarrow a \cup b), (a \cup b), \neg b, (c \Rightarrow b)\}$

1. Show that $G$ is CONSISTENT under classical semantics. Use shorthand notation.

Solution: We find a restricted model for $G$. The formula $(a \Rightarrow a \cup b)$, hence any $v$ is its model. $\neg b = T$ only if $b = F$. We evaluate $(a \cup b) = (a \cup F) = T$ only if $a = T$. Consequently, $(c \Rightarrow b) = (c \Rightarrow F) = T$ only if $c = F$. Hence, any $v$, such that $a = T$, $b = T$, and $c = F$ is a model for $G$. 

1
2. Find a formula $A$ that is iINDEPENDENT of $\mathcal{G}$. Must prove it. Use shorthand notation.

**Solution:** THIS IS MY SOLUTION. THERE ARE MANY OTHERS!

Let $A$ be any atomic formula $d \in \text{VAR} - \{a, b, c\}$. Any $v$, such that $a=T$, $b= T$, and $c= F$, $d= T$ is a model for $\mathcal{G} \cup \{A\}$. Any $v$, such that $a=T$, $b= T$, and $c= F$, $d= F$ is a model for $\mathcal{G} \cup \{\neg A\}$.

3. Find an infinite number of formulas that are iINDEPENDENT of $\mathcal{G}$. Justify your answer.

**Solution:** THIS IS MY SOLUTION. THERE ARE MANY OTHERS!

There is countably infinitely many atomic formulas $A=d$ where $d \in \text{VAR} - \{a, b, c\}$.

**PROBLEM 2**

Given a language $\mathcal{L} = \mathcal{L}_{(\wedge, \lor, \top)}$. We a define a $\mathbf{L}_4$ semantics as follows.

Logical values are $\bot, \bot_1, \bot_2, T$ and they are ordered: $\bot < \bot_1 < \bot_2 < T$.

The connectives are defined as follow

$\neg \bot_1 = \bot_1$, $\neg \bot_2 = \bot_2$, $\neg \bot = T$, $\neg T = F$.

For any $x, y \in \{\bot, \bot_1, \bot_2, T\}$, $x \land y = \min\{x, y\}$, $x \lor y = \max\{x, y\}$, and

$$x \Rightarrow y = \begin{cases} \neg x \lor y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

1. Write Truth Tables for implication and negation.

**Solution:**

\[
\begin{array}{cccc}
\Rightarrow & \bot & \bot_1 & \bot_2 & T \\
F & T & T & T & T \\
\bot_1 & \bot_1 & T & T & T \\
\bot_2 & \bot_2 & \bot_2 & T & T \\
T & T & \bot_1 & \bot_2 & T \\
\end{array}
\]

\[
\neg \begin{array}{cccc}
\neg & \bot & \bot_1 & \bot_2 & T \\
F & T & \bot_1 & \bot_2 & F \\
\end{array}
\]

2. Prove $\not\models_{\mathbf{L}_4}(a \Rightarrow b) \Rightarrow (\neg a \land b)$. Use shorthand notation.

**Solution:** let $v$ be a truth assignment such that $v(a) = v(b) = \bot_1$

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \land b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg \bot_1 \land \bot_1)) = (\bot \Rightarrow (\bot_1 \land \bot_1)) = (\neg \bot_1) = \bot_1$.

This proves that $v$ is a counter-model for our formula and that $\not\models_{\mathbf{L}_4}(a \Rightarrow b) \Rightarrow (\neg a \land b)$.

Observe that there are other counter-models. For example, $v$ such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \land b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg \bot_2 \land \bot_2)) = (\bot \Rightarrow (\bot_2 \land \bot_2)) = (\neg \bot_2) = \bot_2$.

3. Prove that the equivalence defining $\lor$ in terms of negation and implication in classical logic does not hold under $\mathbf{L}_4$, i.e. prove that $(A \lor B) \not\models_{\mathbf{L}_4}(\neg A \Rightarrow B)$.

**Solution:** any $v$ such that $v^*(A) = \bot_2$ and $v^*(B) = \bot_1$ is a counter-model. This is not the only counter-model.

**PROBLEM 3**

Consider the Hilbert system $H1 = (\mathcal{L}_{(\Rightarrow)}, \mathcal{F}, \{A1, A2\}, \{MP, \frac{A \Rightarrow B}{\frac{A}{B}}\})$ where for any $A, B \in \mathcal{F}$

$A1: (A \Rightarrow (B \Rightarrow A))$, $A2: ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$. 

2
1. We have proved that the Deduction Theorem holds for H1.

Use Deduction Theorem to prove \((A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))\).

Solution

We apply the Deduction Theorem twice, i.e. we get

\((A \Rightarrow (C \Rightarrow B)) \vdash_H (C \Rightarrow (A \Rightarrow B))\) if and only if

\((A \Rightarrow (C \Rightarrow B))\), \(C \vdash_H (A \Rightarrow B)\) if and only if

\((A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B\)

We now construct a proof of \((A \Rightarrow (C \Rightarrow B)), C, A \vdash_H B\) as follows

\[B_1 : (A \Rightarrow (C \Rightarrow B))\] hypothesis
\[B_2 : C\] hypothesis
\[B_3 : A\] hypothesis
\[B_4 : (C \Rightarrow B)\] \(B_1, B_3\) and (MP)
\[B_5 : C, B_2, B_4\) and (MP)

2. Explain why 1. proves that \((-a \Rightarrow ((b \Rightarrow -a) \Rightarrow b)) \vdash_H ((b \Rightarrow -a) \Rightarrow (-a \Rightarrow b))\).

Solution This is 1. for \(A = -a\), \(C = (b \Rightarrow -a)\), and \(B = b\).

3. H1 is sound under classical semantics. Explain why H1 is not complete.

Solution The system S is not complete under classical semantics means that not all classical tautologies have a proof in S. We have proved that one needs negation and one of other connectives \(\cup, \cap, \Rightarrow\) to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can’t express negation in terms of implication alone and hence we can’t provide a proof of any tautology i.e. its logically equivalent form in our language \(L_{(\Rightarrow)}\).

4. Let H2 be the proof system obtained from the system H1 by extending the language to contain the negation \(-\) and adding one additional axiom:

\[A3 (\neg(B \Rightarrow A) \Rightarrow (\neg B \Rightarrow A) \Rightarrow B))\]

Explain shortly why Deduction Theorem holds for H2.

Solution The proof of the Deduction Theorem for H1 used only axioms A1, A2 so Adding axiom A3 (and adding \(-\) to the language ) does not change anything in the proof. Hence Deduction Theorem holds for H2.

5. We know that H2 is complete.

Let H3 be the proof system obtained from the system H2 adding additional axiom

\[A4 (\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow B))\]

Does Deduction Theorem holds for S2? Justify.
Solution Yes, it does, by the same argument as for H2.

Does Completeness Theorem holds for S2? Justify.

Solution

No, it doesn’t. The system S2 is not sound. Axiom A4 is not a tautology.

Any v such that A=T and B=F is a counter model for \((\neg (A \Rightarrow B) \Rightarrow \neg (A \Rightarrow \neg B))\).

PROBLEM 4

1 Use the proof system RS and its Completeness Theorem to prove that

\[ \models ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c) \]

Solution

We construct the decomposition tree of A as follows

\[ T_A \]

\[ ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c) \]

\[ | (\cup) \]

\[ ((a \cup b) \Rightarrow \neg a), (\neg a \Rightarrow \neg c) \]

\[ | (\Rightarrow) \]

\[ \neg (a \cup b), \neg a, (\neg a \Rightarrow \neg c) \]

\[ \wedge (\neg \cup) \]

\[ \neg a, \neg a, (\neg a \Rightarrow \neg c) \]

\[ | (\Rightarrow) \]

\[ \neg b, \neg a, (\neg a \Rightarrow \neg c) \]

\[ | (\Rightarrow) \]

\[ \neg a, \neg a, \neg \neg a, \neg c \]

\[ | (\neg \neg) \]

\[ \neg b, \neg a, \neg \neg a, \neg c \]

\[ | (\neg \neg) \]

\[ \neg a, \neg a, a, \neg c \]

\[ \neg b, \neg a, a, \neg c \]

All leaves are axioms, so A has a proof in RS.

We know that RS is complete, so it proves the

\[ \models ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c) \]
2. Use the proof system RS to construct a counter model for the formula

\[((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)\].

**Solution** We construct the decomposition tree of A as follows

\[
\begin{align*}
\text{T}_A & \quad (\((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)\)) \\
| \quad (\cup) & \quad (\((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)\)) \\
\land (\cap) & \quad \neg c, (a \Rightarrow c) \\
(a \Rightarrow b), (a \Rightarrow c) & \quad | (\Rightarrow) \\
| (\Rightarrow) & \quad \neg c, \neg a, c \\
\neg a, b, (a \Rightarrow c) & \quad | (\Rightarrow) \\
\neg a, b, \neg a, c
\end{align*}
\]

The non-axiom leaf is \(L_A : \neg a, b, \neg a, c\).

Any truth assignment \(v : VAR \rightarrow \{T, F\}\) such that \(v(a) = T, v(b) = F, v(c) = F\) falsifies the leaf \(L_A\).

By the strong soundness of rules of RS, \(v\) falsifies the formula at the root of the decomposition tree \(T_A\), i.e. we proved that

\[\not\models (\((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)\)]