CSE371
Some Practice Questions for Midterm 2
SOLUTIONS

QUESTION 1 Let **GL** be the Gentzen style proof system for classical logic defined in chapter 6. Prove, by constructing a proper decomposition tree that

(1) $\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

Solution By definition we have that

$$\vdash_{\mathbf{GL}}((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \ \ if\ and\ only\ if$$

$$\vdash_{\mathbf{GL}} \ \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

$$\mathbf{T}_{\to A}$$

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\to\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\to\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\to \neg)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$\longrightarrow \ \neg b, b, a$	$b \longrightarrow b, a$
$\mid (\to \neg)$	*axiom*
$b \ \longrightarrow \ b, a$	
axiom	

All leaves of the tree are axioms, hence we have found the proof of A in **GL**.

(2) Let **GL** be the Gentzen style proof system defined in chapter 6. Prove, by constructing a proper decomposition tree that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution Observe that for any formula A, its decomposition tree $\mathbf{T}_{\rightarrow A}$ in **GL** is not unique. Hence when constructing decomposition trees we have to cover all possible cases.

We construct the decomposition tree for $\longrightarrow A$ as follows.

$$\mathbf{T_{1 \rightarrow A}}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$(first\ of\ two\ choices)$$

$$\neg b, (a \Rightarrow b) \ \longrightarrow \ a$$

$$\mid (\neg \rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$(one\ choice)$$

$$\longrightarrow \ a, b, a \qquad\qquad\qquad b \longrightarrow b, a$$

$$non-axiom \qquad\qquad\qquad axiom$$

The tree contains a non- axiom leaf $\longrightarrow \ a, b, a$, hence it is not a proof in **GL**. We have only one more tree to construct. Here it is.

$$\mathbf{T_{2 \rightarrow A}}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(one\ choice)$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$(second\ of\ two\ choices)$

$$\begin{array}{ccc}
\longrightarrow\ (\neg b \Rightarrow a), a & \qquad & b \longrightarrow\ (\neg b \Rightarrow a) \\
(\longrightarrow \Rightarrow) & & \mid (\rightarrow \Rightarrow) \\
(one\ choice) & & (one\ choice) \\
\neg b \longrightarrow\ a, a & & b, \neg b \longrightarrow\ a \\
\mid (\neg \rightarrow) & & \mid (\neg \rightarrow) \\
(one\ choice) & & (one\ choice) \\
\longrightarrow\ a, a, b & & b\ \longrightarrow\ a, b \\
non - axiom & & axiom
\end{array}$$

All possible trees end with an non-axiom leave whet proves that $\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

QUESTION 2 Does the tree below constitute a proof in **GL**? Justify your answer.

$$\mathbf{T}_{\rightarrow A}$$

$$\begin{array}{c}
\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
\mid (\rightarrow \neg) \\
\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \\
\mid (\neg \rightarrow) \\
\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \\
\mid (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a) \\
\mid (\rightarrow \Rightarrow) \\
(\neg a \Rightarrow b), \neg b \longrightarrow a \\
\mid (\neg \rightarrow) \\
(\neg a \Rightarrow b) \longrightarrow b, a \\
\bigwedge (\Rightarrow \longrightarrow)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow \neg a, b, a & \qquad\qquad & b \longrightarrow b, a \\
\mid (\rightarrow \neg) & & axiom \\
a \longrightarrow b, a & & \\
axiom & &
\end{array}$$

Solution: The tree is a not a proof in **GL** because a rule corresponding to the decomposition step below does not exists in it.

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

It is a proof is some system **GL1** that has all the rules of **GL** except its $(\neg \rightarrow)$.

This rule has to be replaced by the rule:

$$(\neg \rightarrow)_1 \; \frac{\Gamma, \Gamma' \;\longrightarrow\; \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \;\longrightarrow\; \Delta, \Delta'}.$$

Observe that the completeness of the system **GL** may not imply the completeness of **GL1**, i.e. we don't know if the new system **GL1** is complete (in fact, it is! but has to be proved).

QUESTION 3 Let **GL** be the Gentzen style proof system for classical logic defined in chapter 6. Prove, by constructing a counter-model defined by a proper decomposition tree that

$$\not\models ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b))).$$

Solution

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow (\neg b \cap a)) \Rightarrow (\neg b \Rightarrow (a \cup b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow (\neg b \Rightarrow (a \cup b))$$

$$\mid (\rightarrow\Rightarrow))$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow (a \cup b))$$

$$\mid (\rightarrow \cup)$$

$$one\ of\ two\ choices$$

$$\neg b, (a \Rightarrow (\neg b \cap a)) \longrightarrow a, b$$

$$\mid (\neg \rightarrow)$$

$$(a \Rightarrow (\neg b \cap a)) \longrightarrow b, a, b$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\begin{array}{ccc}
\longrightarrow \neg a, b, a, b & & \longrightarrow (\neg b \cap a), b, a \\
\mid (\rightarrow \neg) & & \bigwedge(\longrightarrow \cap) \\
a \longrightarrow b, a, b & & \\
axiom & \longrightarrow \neg b, b, a, b & \longrightarrow a, b, a, b \\
 & \mid (\rightarrow \neg) & non-axiom \\
 & b \longrightarrow b, a, b & \\
 & axiom &
\end{array}$$

The counter-model model determined by the non-axiom leaf $\longrightarrow a, b, a, b$ is any truth assignment that evaluates it to F.

Observe that (we use a shorthand notation) $\longrightarrow a, b, a, b$ represents semantically $T \longrightarrow a, b, a, b$ and hence $\longrightarrow a, b, a, b = F$ iff $T \longrightarrow a, b, a, b = F$, what happens only if $T \Rightarrow a \cup b \cup a \cup b = F$, i.e when $a = F$ and $b = F$.

QUESTION 4 Prove the COMPLETENESS theorem for GL. Assume that the Soundness has been already proved and the Decompositions Trees are already defined.

Solution

Formula Completeness for GL: for any $A \in \mathcal{F}$,

$$\models A \;\; iff \;\; \vdash_{GL} \rightarrow A$$

Soundness for GL: for any $A \in \mathcal{F}$,

$$If \vdash_{GL} \rightarrow A, \;\; then \models A$$

Completeness part for GL: for any $A \in \mathcal{F}$,

$$If \;\models A, \; then \vdash_{GL} \rightarrow A$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$,

$$If \nvdash_{GL} \rightarrow A \; then \; \not\models A,$$

proof Assume $\nvdash_{GL} \rightarrow A$, i.e. $\rightarrow A$ does not have a proof in GL. Let $\mathcal{T}_A$ be a set of all decomposition trees of $\rightarrow A$. As $\nvdash_{GL} \rightarrow A$, each $T \in \mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_A$. Let $\Gamma' \rightarrow \Delta', \Gamma', \Delta' \in VAR^*$ be an non-axiom leaf of T_A. The non-axiom leaf $\Gamma' \rightarrow \Delta'$ *defines* a truth assignment $v : VAR \rightarrow \{T, F\}$ which falsifies A, as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma' \\ F & \text{if a appears in } \Delta' \end{cases}$$

BY the STRONG soundness of the rules of inference of GL it proves that $\not\models A$.

QUESTION 5 Let **LI** be the Gentzen system for intuitionistic logic as defined in chapter 7.
Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: Observe that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad iff \quad \vdash_{\mathbf{LI}} \longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad .$$

Consider the following decomposition tree $\mathbf{T}_{\rightarrow A}$ of $\rightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$ in **LI**.

$\mathbf{T}_{\rightarrow A}$

$$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow \neg)$$
$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (contr \rightarrow)$$
$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow\Rightarrow)$$
$$(\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\rightarrow\Rightarrow)$$
$$\neg b, (\neg a \Rightarrow b), \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$
$$\mid (exch \rightarrow)$$
$$(\neg a \Rightarrow b), \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow a$$
$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow \neg a$$
$$\mid (\rightarrow \neg)$$
$$a, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (exch \rightarrow)$$
$$a, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), \neg b \longrightarrow$$
$$\mid (exch \rightarrow)$$
$$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)), a, \neg b \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$a, \neg b \longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow\Rightarrow)$$
$$(\neg a \Rightarrow b), a, \neg b \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\rightarrow\Rightarrow)$$
$$\neg b, (\neg a \Rightarrow b), a, \neg b \longrightarrow a$$
$$axiom$$

$$b, \neg b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (exch \rightarrow)$$
$$\neg b, b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$b, \neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow b$$
$$axiom$$

All leaves of $\mathbf{T}_{\rightarrow A}$ are axioms, we have hence found a proof.

QUESTION 6 We know that the formulas below are not Intuitionistic Tautologies. Verify whether **H** semantics (chapter 3) provides a counter-model for them.

$$((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$
$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution

First Formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

We evaluate:$((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = \perp$ iff $(a \Rightarrow b) = T$ and $(\neg a \cup b) = \perp$. Observe that $(\neg a \cup b) = \perp$ in 3 cases, two of which for $\neg a = \perp$ are impossible. We have hence only one case to consider: $\neg a = F, b = \perp$, i.e. $a = \perp$ or $a = T$ and $b = \perp$. Both of them provide a counter-model.

item[] Second formula:

$$((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a)) = \perp$ iff $(\neg a \Rightarrow \neg b) = T$ and $(b \Rightarrow a) = \perp$. The case $(b \Rightarrow a) = \perp$ holds iff $b = T$ and $a = \perp$. In this case $(\neg a \Rightarrow \neg b) = (\neg \perp \Rightarrow \neg T) = F \Rightarrow F = T$. We have a counter-model.

QUESTION 7 Show that

$$\vdash_{\mathbf{LI}} \neg\neg((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$$

Solution We did work it out in class.

QUESTION 8 Use the heuristic method defined in chapter 7 to prove that

$$\nvdash_{\mathbf{LI}} ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)).$$

Solution: To prove that our formula is not provable in **LI** we construct its possible decomposition trees following our heuristic, discuss their relationship and show that each of them must have an non-axiom leaf.

First tree is as follows.

T1

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), \neg b, \longrightarrow a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$\neg b \longrightarrow \neg a$	$b, \neg b \longrightarrow a$
$\mid (\rightarrow \neg)$	$\mid (exch \rightarrow)$
$a, \neg b \longrightarrow$	$\neg b, b \longrightarrow a$
$\mid (exch \rightarrow)$	$\mid (\rightarrow weak)$
$\neg b, a \longrightarrow$	$\neg b, b \longrightarrow$
$\mid (\neg \rightarrow)$	$\mid (\neg \rightarrow)$
$a \longrightarrow b$	$b \longrightarrow b$
$non - axiom$	$axiom$

Second tree The second choice of decomposition rule at the second node of the tree **T1** gives the following tree.

T2

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\longrightarrow \neg a$$
$$\mid (\rightarrow \neg)$$
$$a \longrightarrow$$
$$non - axiom$$

$$b \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\rightarrow\Rightarrow)$$
$$b, \neg b \longrightarrow a$$
$$\mid (exch \rightarrow)$$
$$\neg b, b \longrightarrow a$$
$$\mid (\rightarrow weak)$$
$$\neg b, b \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$b \longrightarrow b$$
$$axiom$$

Observe that **T1** and **T2** have identical sub-trees ending with identical leaves.

Third tree is obtained by the third choice of the decomposition rule at the second node of the tree **T1**, namely the use of rule ($contr \rightarrow$). This step produces a node

$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

Observe that next decomposition steps would give trees similar to **T1** and **T2**. We write down, as an example one of them, which follows the pattern of the tree **T1**.

T3

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$
$$\mid (\rightarrow\Rightarrow)$$
$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (contr \rightarrow)$$
$$(\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$
$$\mid (\rightarrow\Rightarrow)$$
$$\neg b, (\neg a \Rightarrow b), (\neg a \Rightarrow b) \longrightarrow a$$
$$\mid (exch \rightarrow)$$
$$(\neg a \Rightarrow b), \neg b, (\neg a \Rightarrow b) \longrightarrow a$$
$$\bigwedge(\Rightarrow\longrightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (exch \rightarrow)$$

$$\neg b, a, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$a, (\neg a \Rightarrow b) \longrightarrow b$$

$$\mid (exch \rightarrow)$$

$$(\neg a \Rightarrow b), a \longrightarrow b$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$b, \neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (exch \rightarrow)$$

$$\neg b, b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow weak)$$

$$\neg b, b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, (\neg a \Rightarrow b) \longrightarrow b$$

axiom

$$a \longrightarrow \neg a \qquad\qquad b, a \longrightarrow b$$

$$\mid (\rightarrow \neg) \qquad\qquad axiom$$

$$a, a \longrightarrow$$

$$non - axiom$$

Observe that the rule ($contr \rightarrow$) didn't and will never bring information to the tree construction which would replace a non-axiom leaf by an axiom leaf.

Next tree can be obtained by exploring second choice at the node 3 of the first tree.

T4

$$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow weak)$$

$$\neg b, (\neg a \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\begin{array}{ccc}
\longrightarrow \neg a & \qquad\qquad & b \longrightarrow b \\
\mid (\rightarrow \neg) & & axiom \\
a \longrightarrow & & \\
non - axiom & &
\end{array}$$

Observe that here again the rule (*contr* $\rightarrow$) applied to any node to the tree **T4** would never gives us a possibility of replacing a non-axiom leaf by an axiom leaf.

Conclusion All possible decomposition trees will always contain a non- axiom leaf what ends the proof.

Language and Meta-Language

We are using the word ”PROOF” in two distinct senses.

In the first sense, we use it as a **formal proof** within a fixed proof system, for example the system **LI** and is represented as a proof tree, or sequence of expressions of the language $\mathcal{L}$ of **LI**.

In the second sense, it also designates certain sequences of sentences of English language (supplemented by some technical terms, if needed) that are supposed to serve as an argument justifying some assertions about the language $\mathcal{L}$, or proof system based on it.

In general, the language we are studying, in this case $\mathcal{L}$, is called an **OBJECT LANGUAGE**.

The language in which we formulate and prove the results about the object language is called **the METALANGUAGE**. The metalanguage might also be formalized and made the object of study, which we would carry in a **meta-metalanguage**.

We use English as our not formalized metalanguage, although, we use only a mathematically weak portion of the English language. The contrast between the language and metalanguage is also present in study for example, a foreign language. In French study class, French is the object language, while the metalanguage, the language we use, is English.

The distinction between **proof** and **meta-proof**, i.e. a proof in the metalanguage, is now clear. We construct (in the metalanguage) a decomposition tree which is *a formal proof* in the object language. By doing so, we prove in the metalanguage, that the proof in the object language exists. Such proof is called *a meta-proof*, and the fact thus proved is called a *meta-theorem*.