

# CSE371 MIDTERM 1 SOLUTIONS Fall 2017

## PART 1: DEFINITIONS

**D1** Given a language  $\mathcal{L}_{\{\Rightarrow, \cup, \cap, \neg\}}$  and a formula  $A$  of this language.

$\models A$  if and only if  $v \models A$  for all **truth assignment**  $v : VAR \rightarrow \{T, F\}$

**D2** Given formula  $A \in \mathcal{F}$  of  $\mathcal{L}_{\{\Rightarrow, \cup, \cap, \neg\}}$ .

Write definition of  $v$  is a restricted model for  $A$ .

A **restricted MODEL** for the formula  $A$  is any function  $w : VAR_A \rightarrow \{T, F\}$  such that  $w^*(A) = T$ , where

$VAR_A$  is the set of all propositional variables appearing in  $A$ .

**D3** Given a proof system  $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$  and an expression  $E \in \mathcal{E}$ .

$\vdash_S E$  if and only if there is a sequence  $E_1, E_2, \dots, E_n$  of expressions from  $\mathcal{E}$ , such that  $n \geq 1$ , and for each  $1 < i \leq n$ , either  $E_i \in LA$  or  $E_i$  is a **direct consequence** of some of the preceding expressions in  $E_1, E_2, \dots, E_n$  by virtue of one of the rules of inference  $r \in \mathcal{R}$ .

**D4** A proof system  $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$  is **complete** under a semantics  $\mathbf{M}$  if and only if the following holds for any expression  $E \in \mathcal{E}$ .

$$\vdash_S E \quad \text{if and only if} \quad \models_M E.$$

**D5** Write definition: A non-empty set  $\mathcal{G} \subseteq \mathcal{F}$  **consistent** under classical semantics.

A non-empty set  $\mathcal{G} \subseteq \mathcal{F}$  of **formulas** is called **consistent** if and only if  $\mathcal{G}$  **has a model**.

We can also say:

$\mathcal{G} \subseteq \mathcal{F}$  is **consistent** if and only if there is a truth assignment  $v$  such that  $v \models \mathcal{G}$ ,

or we say:

$\mathcal{G} \subseteq \mathcal{F}$  is **consistent** if and only if there is  $v$  such that  $v^*(A) = T$  for all  $A \in \mathcal{G}$

## PART 2: PROBLEMS

### PROBLEM 1

Write the following natural language statement:

**One likes to play bridge, or from the fact that the weather is good we conclude the following:  
one does not like to play bridge or one likes not to play bridge**

as a formula of 2 different languages

1. Formula  $A_1 \in \mathcal{F}_1$  of a language  $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$ , where  $\mathbf{L} A$  represents statement "one likes A", "A is liked".

**Solution** We translate our statement into a formula

$A_1 \in \mathcal{F}_1$  of a language  $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$  as follows.

**Propositional Variables:**  $a, b$   
 $a$  denotes statement: *play bridge*,  
 $b$  denotes a statement: *the weather is good*

**Translation 1**

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. Formula  $A_2 \in \mathcal{F}_2$  of a language  $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ .

**Solution** We translate our statement into a formula  $A_2 \in \mathcal{F}_2$  of a language  $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$  as follows.

**Propositional Variables:**  $a, b, c$   
 $a$  denotes statement: *One likes to play bridge*,  
 $b$  denotes a statement: *the weather is good*,  
 $c$  denotes a statement: *one likes not to play bridge*

**Translation 2:**

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

**PROBLEM 2**

Given a formula  $A : \forall x \exists y P(f(x, y), c)$  of the predicate language  $\mathcal{L}$ , and two **model structures**  $\mathbf{M}_1 = (Z, I_1)$ ,  $\mathbf{M}_2 = (N, I_2)$  with the interpretations defined as follows.

$P_{I_1} : = , \quad f_{I_1} : + , \quad c_{I_1} : 0 \quad \text{and} \quad P_{I_2} : > , \quad f_{I_2} : \cdot , \quad c_{I_2} : 0.$

1. Show that  $\mathbf{M}_1 \models A$

**Solution**

$A_{I_1} : \forall x \in Z \exists y \in Z x + y = 0$  is a **true** statement;  
For each  $x \in Z$  exists  $y = -x$  and  $-x \in Z$  and  $x - x = 0$ .

**Solution**

2. Show that  $\mathbf{M}_2 \not\models A$ .

**Solution**

$A_{I_2} : \forall x \in N \exists y \in N x \cdot y > 0$  is a **false** statement for  $x = 0$ .

**PROBLEM 3**

We define a 3 valued extensional semantics  $\mathbf{M}$  for the language  $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$  by **defining the connectives**  $\neg, \cup, \Rightarrow$  on a set  $\{F, \perp, T\}$  of logical values by the following truth tables.

**L Connective**

<b>L</b>	F	$\perp$	T
	F	F	T

**Negation :**

$\neg$	F	$\perp$	T
	T	F	F

**Implication**

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	F	F	T

**Disjunction :**

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	T	T

1. Verify whether  $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$ . You can use **shorthand notation**.

**Solution**

We verify

$$\mathbf{L}T \cup \neg \mathbf{L}T = T \cup F = T, \quad \mathbf{L} \perp \cup \neg \mathbf{L} \perp = F \cup \neg F = F \cup T = T, \quad \mathbf{L}F \cup \neg \mathbf{L}F = F \cup \neg F = T$$

2. Verify whether your formulas  $A_1$  and  $A_2$  from PROBLEM 1 have a model/ counter model under the semantics **M**. You can use **shorthand notation**.

**Solution**

The formulas are:  $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$ , and  $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$ .

Any  $v$ , such that  $v(a) = T$  is a **M model** for  $A_1$  and for  $A_2$  directly from the definition of  $\cup$ .

3. Verify whether the following set **G** is **M-consistent**. You can use **shorthand notation**

$$\mathbf{G} = \{ \mathbf{L}a, (a \cup \neg \mathbf{L}b), (a \Rightarrow b), b \}$$

**Solution**

Any  $v$ , such that  $v(a) = T, v(b) = T$  is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg \mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.$$

**PROBLEM 4**

Let  $S$  be the following **proof system**

$$S = ( \mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}, \mathcal{F}, \{ \mathbf{A1}, \mathbf{A2} \}, \{ (r1), (r2) \} )$$

for the logical axioms and rules of inference defined for any formulas  $A, B \in \mathcal{F}$  as follows

**Logical Axioms**

$$\mathbf{A1} \quad (\mathbf{L}A \cup \neg \mathbf{L}A)$$

$$\mathbf{A2} \quad (A \Rightarrow \mathbf{L}A)$$

**Rules** of inference:

$$(r1) \quad \frac{A ; B}{(A \cup B)}, \quad (r2) \quad \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. Verify whether the system  $S$  is **M-sound**. You can use **shorthand notation**

**Solution**

**Observe** that both logical axioms of  $S$  are **M tautologies**

**A1, A2** are **M tautologies** by direct evaluation.

Rule (r1) is **sound** because when  $A = T$  and  $B = T$  we get  $A \cup B = T \cup T = T$ .

Rule (r2) is **not sound** because when  $A = T$  and  $B = F$  (or  $B = \perp$ ) we get  $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$  or  $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

We proved that  $S$  is **not sound**.

**2.** Show, by constructing a proper **formal proof** that

$$\vdash_S ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$$

You must write comments how each step of the proof was obtained

Write all steps of the **formal proof** as follows - write as MANY as you NEED!

**Solution**

Here is the proof  $B_1, B_2, B_3, B_4$  of  $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$ .

$B_1$ :  $(\mathbf{L}a \cup \neg \mathbf{L}a)$  Axiom  $A_1$  for  $A = a$

$B_2$ :  $\mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)$  rule (r2) for  $B = b$  applied to  $B_1$

$B_3$ :  $(\mathbf{L}b \cup \neg \mathbf{L}b)$  Axiom  $A_1$  for  $A = b$

$B_4$ :  $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$  (r1) applied to  $B_3$  and  $B_2$ .

**3.** Does above point **2.** prove that  $\models ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$ ?

**Solution**

**No**, it doesn't because the system  $S$  is **not sound**

**PROBLEM 5**

Consider the Hilbert system  $H_1 = (\mathcal{L}_{\{\Rightarrow\}}, \mathcal{F}, \{A1, A2\}, (MP) \frac{A; (A \Rightarrow B)}{B})$  where

$A1$ :  $(A \Rightarrow (B \Rightarrow A))$ ,  $A2$ :  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$  and  $A, B$  are any formulas from  $\mathcal{F}$ .

Use **Deduction Theorem** to prove  $\vdash_{H_1} ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ .

Write comments how each step was obtained.

**Solution**

By Deduction Theorem applied THREE times we get that

$\vdash_{H_1} ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$  if and only if

$(B \Rightarrow C) \vdash_{H_1} ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$  if and only if

$(B \Rightarrow C), (A \Rightarrow B) \vdash_{H_1} (A \Rightarrow C)$  if and only if

$(B \Rightarrow C), (A \Rightarrow B), A \vdash_{H_1} C$ .

Here is a formal proof of  $C$  from  $(A \Rightarrow B), (B \Rightarrow C), A$ :

$B_1 \quad (A \Rightarrow B) \quad \text{Hyp}$

$B_2 \quad A \quad \text{Hyp}$

$B_3 \quad B \quad \text{MP on } B_2, B_1$

$B_4 \quad (B \Rightarrow C) \quad \text{Hyp}$

$B_5 \quad C \quad \text{MP on } B_3, B_4$