# cse371/mat371
# LOGIC

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# LECTURE 6

# Chapter 6
# Automated Proof Systems for Classical Propositional Logic

PART 1: **RS** SYSTEM

PART 2: **RS1, RS2, RS3** SYSTEMS

PART 3: GENTZEN SYSTEMS

# CLASSICAL AUTOMATED PROOF SYSTEMS

Hilbert style systems are easy to define and admit a relatively simple proofs of the Completeness Theorem but they are difficult to use

Automated systems are less intuitive then the Hilbert-style systems, but they will allow us to define effective automatic procedures for **proof search**, what is impossible in a case of the Hilbert-style systems

The first idea of this type was presented by **G. Gentzen** in 1934

We present in this chapter our version of original Gentzen system for **propositional classical logic**

We present the original Gentzen systems for **Intuitionistic** and **Classical** Propositional Logics in Chapter 7

# AUTOMATED PROOF SYSTEMS

## PART 1: RS System

The automated proof system we presented here is due to Helena Rasiowa and Roman Sikorski

We present here the propositional version of the original system and call it RS system for Rasiowa - Sikorski

The propositional RS system extends naturally to predicate logic QRS system which is presented in Chapter 10

Both systems RS and QRS admit a **constructive proof** of **Completeness Theorem**

First such constructive proofs were given, together with the formalization of the systems by H. Rasiowa and Sikorski in 1961

AUTOMATED PROOF SYSTEMS

PART 2: RS1, RS2 Systems

We define, as an exercise 2 versions of of the RS System, discuss their differences and show how the proof of **Completeness Theorem** for RS **extends** to similar proofs for all 3 systems

# AUTOMATED PROOF SYSTEMS

PART 3: GENTZEN Systems - Lecture 6a

We present our modern versions of Gentzen Sequent systems for propositional classical logic

Both systems **extend** easily to predicate logic and admit a **constructive proof** of **Completeness Theorem** via Rasiowa-Sikorski method

We also present the original Gentzen systems **LK** for classical propositional logic together with the original Gentzen proof of Hauptzatz ( Cut Elimination Theorem)

## PART1:
## RS Proof System for Classical Propositional Logic

# RS Proof System

Language of **RS** is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

The rules of inference of our system **RS** operate on **finite sequences** of formulas and we adopt

$$\mathcal{E} = \mathcal{F}^*$$

as the set of expressions of **RS**

**Notation**

Elements of $\mathcal{E}$ are finite sequences of formulas and we denote them by

$$\Gamma, \Delta, \Sigma \dots$$

with indices if necessary.

# RS Proof System

The the **intuitive meaning** of a sequence $\Gamma \in \mathcal{F}^*$ is that the truth assignment $v$ makes it **true** if and only if it makes the formula of the form of the disjunction of all formulas of $\Gamma$ **true**

For any sequence $\Gamma \in \mathcal{F}^*$,

$$\Gamma = A_1, A_2, ..., A_n$$

we **denote**

$$\delta_\Gamma = A_1 \cup A_2 \cup ... \cup A_n$$

We define as the next step a **formal semantics** for **RS**

# Formal Semantics for RS

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment and

$v^*$ its classical semantics extension to the set of formulas $\mathcal{F}$

We formally **extend** $v$ to the set $\mathcal{F}^*$ of all finite sequences of $\mathcal{F}$ as follows

$$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(A_1) \cup v^*(A_2) \cup ... \cup v^*(A_n)$$

The sequence $\Gamma$ is said to be **satisfiable** if there is a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that $v^*(\Gamma) = T$

We write it as

$$v \models \Gamma$$

and call $v$ a **model** for $\Gamma$

## Formal Semantics for RS

The sequence $\Gamma$ is said to be **falsifiable** if there is a truth assignment $v$, such that $v^*(\Gamma) = F$

Such a truth assignment $v$ is called a **counter-model** for $\Gamma$

The sequence $\Gamma$ is said to be a **tautology** iff $v^*(\Gamma) = T$ for all truth assignments $v : VAR \longrightarrow \{T, F\}$

We write as always,

$$\models \Gamma$$

to denote that $\Gamma$ is a **tautology**

# Example

**Example**

Let $\Gamma$ be a sequence

$$a, (b \cap a), \neg b, (b \Rightarrow a)$$

The truth assignment $v$ such that

$$v(a) = F \quad \text{and} \quad v(b) = T$$

**falsifies** $\Gamma$, i.e. is a **counter-model** for $\Gamma$ as shows the following computation

$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(a) \cup v^*(b \cap a) \cup v^*(\neg b) \cup v^*(b \Rightarrow a) = F \cup (F \cap T) \cup F \cup (T \Rightarrow F) = F \cup F \cup F \cup F = F.$

# Rules of inference

Rules of inference of **RS** are of the form:

$$\frac{\Gamma_1}{\Gamma} \quad \text{or} \quad \frac{\Gamma_1 \; ; \; \Gamma_2}{\Gamma}$$

where $\Gamma_1, \Gamma_2$ are called **premisses** and $\Gamma$ is called the **conclusion** of the rule

Each rule of inference **introduces** a new logical connective or a negation of a logical connective

We **name** the rule that introduces the logical connective $\circ$ in the conclusion sequent $\Gamma$ by $(\circ)$

**The notation** $(\neg\circ)$ means that the negation of the logical connective $\circ$ is introduced in the conclusion sequence $\Gamma$

# Rules of inference of system bf RS

Proof System **RS** contains seven inference rules:

$$(\cup), \quad (\neg\cup), \quad (\cap), \quad (\neg\cap), \quad (\Rightarrow), \quad (\neg \Rightarrow), \quad (\neg\neg)$$

Before we **define** the rules of inference of **RS** we need to introduce some definitions.

**Definition**

Any propositional variable, or a negation of propositional variable is called a **literal**

The set

$$LT = VAR \cup \{\neg a : \quad a \in VAR\}$$

is called a set of all propositional **literals**

The variables are called **positive literals**

Negations of variables are called **negative literals**.

# Literal

We denote by

$$\Gamma', \quad \Delta', \quad \Sigma' \dots$$

finite sequences (empty included) formed out of **literals** i.e

$$\Gamma', \Delta', \Sigma' \in LT^*$$

We will denote by

$$\Gamma, \quad \Delta, \quad \Sigma \dots$$

the elements of $\mathcal{F}^*$

## Logical Axioms of RS

We adopt as an logical axiom of **RS** any sequence of **literals** which contains a propositional variable and its negation, i.e any sequence

$$\Gamma'_1,\ a,\ \Gamma'_2,\ \neg a,\ \Gamma'_3$$

$$\Gamma'_1,\ \neg a,\ \Gamma'_2,\ a,\ \Gamma'_3$$

where $a \in VAR$ is any **propositional variable**

We denote by LA the set of all logical axioms of **RS**

# Inference Rules of RS

**Disjunction rules**

$$(\cup)\ \frac{\Gamma',\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta}, \qquad (\neg\cup)\ \frac{\Gamma',\ \neg A,\ \Delta\ \ ;\ \ \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cup B),\ \Delta}$$

**Conjunction rules**

$$(\cap)\ \frac{\Gamma',\ A,\ \Delta\ \ ;\ \ \Gamma',\ B,\ \Delta}{\Gamma',\ (A \cap B),\ \Delta}, \qquad (\neg\cap)\ \frac{\Gamma',\ \neg A,\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cap B),\ \Delta}$$

# Inference Rules of RS

**Implication rules**

$$(\Rightarrow)\ \frac{\Gamma',\ \neg A, B,\ \Delta}{\Gamma',\ (A \Rightarrow B),\ \Delta}, \qquad (\neg\Rightarrow)\ \frac{\Gamma',\ A,\ \Delta \quad : \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A \Rightarrow B),\ \Delta}$$

**Negation rule**

$$(\neg\neg)\ \frac{\Gamma',\ A,\ \Delta}{\Gamma',\ \neg\neg A,\ \Delta}$$

where $\Gamma' \in LT^*,\ \ \Delta \in \mathcal{F}^*,\ \ A, B \in \mathcal{F}$

# Proof System RS

Formally we define the system **RS** as follows

$$\textbf{RS} = (\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}},\ \mathcal{F}^*,\ LA,\ \mathcal{R})$$

where the set of inference rules is

$$\mathcal{R} = \{(\cup),\ (\neg\cup),\ (\cap),\ (\neg\cap),\ (\Rightarrow),\ (\neg\Rightarrow),\ (\neg\neg)\}$$

and LA is the set of all logical axioms

## Proof Trees

**Definition**

By a **proof tree** in **RS** of $\Gamma$ we understand a tree

$$\mathbf{T}_{\Gamma}$$

built out of sequences satisfying the following conditions:

**1.** The topmost sequence, i.e the root of $\mathbf{T}_{\Gamma}$ is the sequence $\Gamma$

**2.** all leafs are axioms

**2.** the nodes are sequences such that each sequence on the tree follows from the ones immediately preceding it by one of the inference rules

# Proof Trees

We picture, and write our proof trees with the **root** on the top, and the **leafs** on the very bottom,

**Additionally** we write our proof trees indicating the name of the inference rule used at each step of the proof

**Example**

Assume that a **proof** of a sequence $\Gamma$ from some three axioms was obtained by the subsequent use of the rules $(\cap), (\cup), (\cup), (\cap), (\cup)$, and $(\neg\neg)$, $(\Rightarrow)$

We represent it as the following tree

# Proof Trees

## The tree $\mathbf{T}_{\Gamma}$

$\Gamma$

$\mid (\Rightarrow)$

*conclusion of* $(\neg\neg)$

$\mid (\neg\neg)$

*conclusion of* $(\cup)$

$\mid (\cup)$

*conclusion of* $(\cap)$

$\bigwedge (\cap)$

*conclusion of* $(\cap)$ | *conclusion of* $(\cup)$

$\mid (\cup)$ | $\mid (\cup)$

*axiom* | *conclusion of* $(\cap)$

$\bigwedge (\cap)$

*axiom* *axiom*

# Proof Trees

The Proof Trees represent a certain visualization for the proofs

Any formal proof in any proof system can be represented in a tree form and vice- versa

Any proof tree can be re-written in a linear form as a previously defined formal proof

**Example**

The proof tree in **RS** of the de Morgan Law

$$A = (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

is the as follows

# Proof Trees

The tree $\mathbf{T}_A$

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$\mid (\neg\neg)$$

$$(a \cap b), (\neg a \cup \neg b)$$

$$\bigwedge(\cap)$$

$$a, (\neg a \cup \neg b) \qquad\qquad b, (\neg a \cup \neg b)$$

$$\mid (\cup) \qquad\qquad \mid (\cup)$$

$$a, \neg a, \neg b \qquad\qquad b, \neg a, \neg b$$

## Formal Proof

To obtain a formal proof (written in a vertical form) of $A$ it we just write down the tree as a sequence, starting from the leafs and going up (from left to right) to the root

$$a, \neg a, \neg b$$

$$b, \neg a, \neg b$$

$$a, (\neg a \cup \neg b)$$

$$b, (\neg a \cup \neg b$$

$$(a \cap b), (\neg a \cup \neg b)$$

$$\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

**Example**

A search for the proof in **RS** of other de Morgan Law

$$A = (\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

consists of building a certain tree and proceeds as follows.

# Example

The tree $\mathbf{T}_A$

$$(\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(a \cup b), (\neg a \cap \neg b)$$

$$\mid (\neg\neg)$$

$$(a \cup b), (\neg a \cap \neg b)$$

$$\mid (\cup)$$

$$a, b, (\neg a \cap \neg b)$$

$$\bigwedge (\cap)$$

$$a, b, \neg a \qquad\qquad a, b, \neg b$$

# Example

We construct its formal proof , as before, written in a vertical manner

Here it is

$$a, b, \neg b$$

$$a, b, \neg a$$

$$a, b, (\neg a \cap \neg b)$$

$$(a \cup b), (\neg a \cap \neg b)$$

$$\neg\neg(a \cup b), (\neg a \cap \neg b)$$

$$(\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

## Decomposition Trees

Our GOAL in inventing proof systems like **RS** is to facilitatee automatic proof search

The method of such proof search is to generate what is called the **decomposition trees**

The decomposition tree for

$$A = (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

is built as follows

# Decomposition Trees

The tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

# RS: DECOMPOSITION RULES
# and
# DECOMPOSITION TREES

## Decomposition Trees

The process of searching for a proof of a formula $A \in \mathcal{F}$ in **RS** consists of building a certain tree $\mathbf{T}_A$, called a **decomposition tree**

Building a **decomposition tree**, i.e. a proof search tree consists in the **first step** of transforming the **RS rules** into corresponding **decomposition rules**

## RS Decomposition Rules

Here are all of **RS** **decomposition rules**

**Disjunction decomposition rules**

$$(\cup)\ \frac{\Gamma',\ (A \cup B),\ \Delta}{\Gamma',\ A, B,\ \Delta}, \qquad (\neg\cup)\ \frac{\Gamma',\ \neg(A \cup B),\ \Delta}{\Gamma',\ \neg A,\ \Delta \ \ ; \ \ \Gamma',\ \neg B,\ \Delta}$$

**Conjunction decomposition rules**

$$(\cap)\ \frac{\Gamma',\ (A \cap B),\ \Delta}{\Gamma', A, \Delta \ \ ; \ \ \Gamma',\ B, \Delta}, \qquad (\neg\cap)\ \frac{\Gamma',\ \neg(A \cap B),\ \Delta}{\Gamma',\ \neg A, \neg B,\ \Delta}$$

# Decomposition Rules

**Implication decomposition rules**

$$(\Rightarrow)\ \frac{\Gamma',\ (A \Rightarrow B),\ \Delta}{\Gamma',\ \neg A, B,\ \Delta}, \qquad (\neg \Rightarrow)\ \frac{\Gamma',\ \neg(A \Rightarrow B),\ \Delta}{\Gamma', A, \Delta \quad ; \quad \Gamma',\ \neg B,\ \Delta}$$

**Negation decomposition rule**

$$(\neg\neg)\ \frac{\Gamma',\ \neg\neg A,\ \Delta}{\Gamma',\ A,\ \Delta}$$

where $\Gamma' \in \mathcal{F}'^*,\ \ \Delta \in \mathcal{F}^*,\ \ A, B \in \mathcal{F}$

## Tree Decomposition Rules

We write the decomposition rules in a **visual tree form** as follows

**Tree Decomposition Rules**

$(\cup)$ **rule**

$$\Gamma^{'},\ (A \cup B),\ \Delta$$

$$\mid (\cup)$$

$$\Gamma^{'},\ A, B,\ \Delta$$

# Tree Decomposition Rules

$(\neg\cup)$ **rule**

$$\Gamma^{'},\ \neg(A \cup B),\ \Delta$$

$$\bigwedge(\neg\cup)$$

$$\Gamma^{'},\ \neg A,\ \Delta \qquad\qquad \Gamma^{'},\ \neg B,\ \Delta$$

$(\cap)$ **rule**

$$\Gamma^{'},\ (A \cap B),\ \Delta$$

$$\bigwedge(\cap)$$

$$\Gamma^{'},\ A,\ \Delta \qquad\qquad \Gamma^{'},\ B,\ \Delta$$

# Tree Decomposition Rules

**$(\neg\cup)$ rule**

$$\Gamma^{'},\ \neg(A \cap B),\ \Delta$$

$$\mid (\neg\cap)$$

$$\Gamma^{'},\ \neg A, \neg B,\ \Delta$$

**$(\Rightarrow)$ rule**

$$\Gamma^{'},\ (A \Rightarrow B),\ \Delta$$

$$\mid (\cup)$$

$$\Gamma^{'},\ \neg A, B,\ \Delta$$

## Tree Decomposition Rules

$(\neg \Rightarrow)$ **rule**

$$\Gamma', \neg(A \Rightarrow B), \Delta$$

$$\bigwedge(\neg \Rightarrow)$$

$$\Gamma', A, \Delta \qquad\qquad \Gamma', \neg B, \Delta$$

$(\neg\neg)$ **rule**

$$\Gamma', \neg\neg A, \Delta$$

$$\mid (\neg\neg)$$

$$\Gamma', A, \Delta$$

## Definitions and Observations

**Observe** that we use the same names for the **inference** and **decomposition** rules, as once the we have built the decomposition tree with all leaves being axioms, it constitutes a **proof** of $A$ in **RS** with branches labeled by the proper **inference rules**

Now we still need to introduce few standard and useful definitions and observations.

**Definition: Indecomposable Sequence**

A sequence $\Gamma'$ built only out of literals, i.e. $\Gamma \in \mathcal{F}'^{*}$ is called an **indecomposable sequence**

## Definitions and Observations

**Definition: Indecomposable Sequence**

A sequence $\Gamma'$ built only out of literals, i.e. $\Gamma \in \mathcal{F}'^*$ is called an **indecomposable sequence**

**Definition: Decomposable Formula**

A formula A that is not a literal, i.e. $A \in \mathcal{F} - LT$ is called a decomposable formula

**Definition: Decomposable Sequence**

A sequence $\Gamma$ that contains a decomposable formula is called a decomposable sequence

## Definitions and Observations

**Observation 1**

For any **decomposable** sequence, i.e. for any $\Gamma \notin LT^*$ there is **exactly one** decomposition rule that can be applied to it

This rule is **determined** by the first decomposable formula in $\Gamma$ and by the main connective of that formula

# Definitions and Observations

**Observation 2**

If the main connective of the **first** decomposable formula is $\cup, \cap, \Rightarrow$,

then the **decomposition rule** determined by it is $(\cup), (\cap), (\Rightarrow)$, respectively

**Observation 3**

If the main connective of the **first** decomposable formula $A$ is negation $\neg$

then the **decomposition rule** is determined by the **second connective** of the formula $A$

The corresponding **decomposition rules** are $(\neg\cup), (\neg\cap), (\neg\neg), (\neg \Rightarrow)$

## Decomposition Lemma

Because of the importance of the **Observation 1** we re-write it in a form of the following

**Decomposition Lemma**

For any sequence $\Gamma \in \mathcal{F}^*$,

$\Gamma \in LT^*$ or $\Gamma$ is in the domain of **exactly one** of **RS** Decomposition Rules

This rule is **determined** by the first decomposable formula in $\Gamma$ and by the main connective of that formula

## Decomposition Tree Definition

**Definition: Decomposition Tree $\mathbf{T}_A$**

For each $A \in \mathcal{F}$, a **decomposition tree** $\mathbf{T}_A$ is a tree build as follows

**Step 1.**

The formula $A$ is the **root** of $\mathbf{T}_A$

For any other **node** $\Gamma$ of the tree we follow the steps below

**Step 2.**

If $\Gamma$ is **indecomposable** then $\Gamma$ becomes a **leaf** of the tree

# Decomposition Tree Definition

**Step 3.**

If $\Gamma$ is **decomposable**, then we **traverse** $\Gamma$ from **left** to **right** and identify the **first** **decomposable formula** $B$

By the **Decomposition Lemma**, there is exactly one decomposition rule determined by the main connective of $B$

**We put** its premiss as a **node below**, or its left and right premisses as the left and right **nodes below**, respectively

**Step 4.**

We repeat **Step 2** and **Step 3** until we obtain only leaves

## Decomposition Theorem

We now prove the following **Decomposition Tree Theorem**.

This Theorem provides a crucial step in the proof of the Completeness Theorem for RS

**Decomposition Tree Theorem**

For any sequence $\Gamma \in \mathcal{F}^*$ the following conditions hold

**1.** $\mathbf{T}_\Gamma$ is finite and unique

**2.** $\mathbf{T}_\Gamma$ is a proof of $\Gamma$ in **RS** if and only if all its leafs are axioms

**3.** $\nvdash_{\mathbf{RS}} \Gamma$ if and only if $\mathbf{T}_\Gamma$ has a non- axiom leaf

## Theorem

**Proof**

The tree $\mathbf{T}_{\Gamma}$ is unique by the **Decomposition Lemma**

It is finite because there is a finite number of logical connectives in $\Gamma$ and all decomposition rules diminish the number of connectives

If the tree $\mathbf{T}_{\Gamma}$ has a **non- axiom** leaf it is not a proof by definition

By **1.** it also means that the proof does not exist

**Example**

Let's construct, as an example a decomposition tree $\mathbf{T}_A$ of the following formula $A$

$$((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

The formula A forms a one element **decomposable sequence**

The first decomposition rule used is determined by its main connective

We put a **box** around it, to make it more visible

$$((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

## Example

The first and only decomposition rule to be applied is $(\cup)$

The first segment of the decomposition tree $\mathbf{T}_A$ is

$$((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \Rightarrow \neg a), \ (\neg a \Rightarrow \neg c)$$

## Example

Now we decompose the sequence

$$((a \cup b) \Rightarrow \neg a),\ (\neg a \Rightarrow \neg c)$$

It is a **decomposable** sequence with the first, decomposable formula

$$((a \cup b) \Rightarrow \neg a)$$

The next step of the construction of our decomposition tree is determined by its main connective $\Rightarrow$ and we put the box around it

$$((a \cup b) \boxed{\Rightarrow} \neg a),\ (\neg a \Rightarrow \neg c)$$

# Example

The decomposition tree becomes now

$$((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\neg(a \cup b), \neg a, (\neg a \Rightarrow \neg c)$$

## Example

The next sequence to decompose is

$$\neg(a \cup b), \neg a, (\neg a \Rightarrow \neg c)$$

with the first decomposable formula

$$\neg(a \cup b)$$

Its main connective is $\neg$, so to find the appropriate rule we have to examine next connective, which is $\cup$

The **decomposition rule** determine by this stage of decomposition is $(\neg\cup)$

## Example

Next stage of the construction of the decomposition tree $\mathbf{T}_A$ is

$$((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\boxed{\neg}(a \boxed{\cup} b), \neg a, (\neg a \Rightarrow \neg c)$$

$$\bigwedge (\neg \cup)$$

$$\neg a, \neg a, (\neg a \Rightarrow \neg c) \qquad \neg b, \neg a, (\neg a \Rightarrow \neg c)$$

# Example

Finally, the complete $\mathbf{T}_A$ is

$$((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\boxed{\neg} (a \boxed{\cup} b), \neg a, (\neg a \Rightarrow \neg c)$$

$$\bigwedge (\neg \cup)$$

$$\neg a, \neg a, (\neg a \boxed{\Rightarrow} \neg c) \qquad \neg b, \neg a, (\neg a \boxed{\Rightarrow} \neg c)$$

$$\mid (\Rightarrow) \qquad \mid (\Rightarrow)$$

$$\neg a, \neg a, \boxed{\neg\neg} a, \neg c \qquad \neg b, \neg a, \boxed{\neg\neg} a, \neg c$$

$$\mid (\neg\neg) \qquad \mid (\neg\neg)$$

$$\neg a, \neg a, a, \neg c \qquad \neg b, \neg a, a, \neg c$$

# Example

All leaves of $\mathbf{T}_A$ are axioms

The tree $\mathbf{T}_A$ is a **proof** of $A$ in **RS**, i.e.

$$\vdash_{\mathbf{RS}} ((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

# Example

**Example** Given a formula $A$ and its decomposition tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

## Example

There is a leaf $\neg a, b, \neg a, c$ of the tree $\mathbf{T}_A$ that is **not an axiom.** By the **Decomposition Tree Theorem**

$$\nvdash_{\mathbf{RS}} ((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

It means that the proof in **RS** of the formula $((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$ **does not exists**

# Completeness Theorem

Our main goal is to prove the **Completeness Theorem** for **RS**

We **prove** first the following **Completeness Theorem** for formulas $A \in \mathcal{F}$

**Completeness Theorem 1** For any formula $A \in \mathcal{F}$

$$\vdash_{\mathbf{RS}} A \quad \text{if and only if} \quad \models A$$

and then we generalize it to the following

**Completeness Theorem 2** For any $\Gamma \in \mathcal{F}^*$,

$$\vdash_{\mathbf{RS}} \Gamma \quad \text{if and only if} \quad \models \Gamma$$

Do do so we need to introduce a new notion of a Strong Soundness and prove that the **RS** is strongly sound

Strong Soundness of **RS**

## Strong Soundness

**Definition**

Given a proof system

$$S = (\mathcal{L},\ \mathcal{E},\ LA,\ \mathcal{R})$$

**Definition**

A rule $r \in \mathcal{R}$ such that the conjunction of all its premisses is **logically equivalent** to its conclusion is called **strongly sound**

**Definition**

A proof system $S$ is called **strongly sound** iff $S$ is sound and **all** its rules $r \in \mathcal{R}$ are **strongly sound**

# Strong Soundness of RS

**Theorem**

The proof system **RS** is strongly sound

**Proof**

We prove as an example the **strong soundness** of two of inference rules: $(\cup)$ and $(\neg\cup)$

Proof for all other rules follows the same patterns and is left as an exercise

By definition of strong soundness we have to show that If $P_1$, $P_2$ are premisses of a given rule and $C$ is its conclusion, then for all v,

$$v^*(P_1) = v^*(C)$$

in case of one premiss rule and

$$v^*(P_1) \cap v^*(P_2) = v^*(C)$$

in case of the two premisses rule.

## Strong Soundness of RS

Consider the rule $(\cup)$

$$(\cup) \quad \frac{\Gamma',\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta}$$

We evaluate:

$$v^*(\Gamma', A, B, \Delta) = v^*(\delta_{\{\Gamma', A, B, \Delta\}}) = v^*(\Gamma') \cup v^*(A) \cup v^*(B) \cup v^*(\Delta)$$

$$= v^*(\Gamma') \cup v^*(A \cup B) \cup v^*(\Delta) = v^*(\delta_{\{\Gamma', (A \cup B), \Delta\}})$$

$$= v^*(\Gamma', (A \cup B), \Delta)$$

# Strong Soundness of RS

Consider the rule $(\neg\cup)$

$$(\neg\cup)\quad \frac{\Gamma',\ \neg A,\ \Delta \quad : \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A\cup B),\ \Delta}$$

We evaluate:

$$v^*(P_1)\cap v^*(P_2) = v^*(\Gamma', \neg A, \Delta)\cap v^*(\Gamma', \neg B, \Delta)$$

$$= (v^*(\Gamma')\cup v^*(\neg A)\cup v^*(\Delta))\cap(v^*(\Gamma')\cup v^*(\neg B)\cup v^*(\Delta))$$

$$= (v^*(\Gamma', \Delta)\cup v^*(\neg A))\cap(v^*(\Gamma', \Delta)\cup v^*(\neg B))$$

$$=^{distrib}\ (v^*(\Gamma', \Delta)\cup(v^*(\neg A)\cap v^*(\neg B))$$

$$= v^*(\Gamma')\cup v^*(\Delta)\cup v^*(\neg A\cap\neg B) =^{deMorgan}\ v^*(\delta_{\{\Gamma', \neg(A\cup B), \Delta\}}$$

$$= v^*(\Gamma',\ \neg(A\cup B),\ \Delta) = v^*(C)$$

## Soundness Theorem

**Observe** that the strong soundness notion implies soundness (not only by name!). Obviously the LA of **RS** are **tautologies** , hence we have also proved the following

**Soundness Theorem** for **RS**

For any $\Gamma \in \mathcal{F}^*$,

$$\text{If} \quad \vdash_{\mathbf{RS}} \Gamma, \quad \text{then} \quad \models \Gamma$$

In particular, for any $A \in \mathcal{F}$,

$$\text{If} \quad \vdash_{\mathbf{RS}} A, \quad \text{then} \quad \models A$$

# Strong Soundness

We proved that all the rules of inference of **RS** of are **strongly sound**, i.e. $C \equiv P$ and $C \equiv P_1 \cap P_2$

**Strong soundness** of the rules means that if **at least** one of premisses of a rule is **false**, so is its conclusion

Given a formula A, such that its $\mathbf{T}_A$ has a branch ending with a non-axiom leaf

By **strong soundness**, any v that make this non-axiom leaf **false** also **falsifies** all sequences on that branch, and hence **falsifies the** the formula A

This means that any v that **falsifies** a non-axiom leaf is a **counter-model** for A

# Counter Model Theorem

We have proved the following

**Counter Model Theorem**

Let $A \in \mathcal{F}$ be such that its decomposition tree $\mathbf{T}_A$ contains a **non- axiom** leaf $L_A$

Any truth assignment v that **falsifies** $L_A$ is a **counter model** for A

Any truth assignment that falsifies a non- axiom leaf is called a **counter-model** for $A$ **deftermined** by the decomposition tree $\mathbf{T}_A$

# Counter Model Example

Consider a tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) \qquad\qquad \neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow) \qquad\qquad \mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) \qquad\qquad \neg c, \neg a, c$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

## Counter Model Example

The tree $\mathbf{T}_A$ has a **non-axiom leaf** $L_A: \neg a, b, \neg a, c$

We define a truth assignment $v: VAR \longrightarrow \{T, F\}$ that **falsifies** the leaf $L_A$ as follows

Observe that v must be such that

$v^*(\neg a, b, \neg a, c) = v^*(\neg a) \cup v^*(b) \cup v^*(\neg a) \cup v^*(c) =$ $\neg v(a) \cup v(b) \cup \neg v(a) \cup v(c) = F$, i.e. all components of the disjunction must be put to F

We hence get that v must be such that

$$v(a) = T, \quad v(b) = F, \quad v(c) = F$$

By the **Counter Model Theorem**, the v **determined** by the non-axiom leaf also **falsifies** the formula A, i.e. we proved that v is a **counter model** for A and

$$\not\models (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

# Counter Model

The **Counter Model Theorem** says that **F** determined by the non-axiom leaf "climbs" the tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)) = \mathbf{F}$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c) = \mathbf{F}$$

$$\bigwedge(\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c = \mathbf{F}$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

*axiom*

# Counter Model

**Observe** that the same counter model construction applies to any other non-axiom leaf, if exists

The other non-axiom leaf defines another **F** that also "climbs the tree" picture, and hence defines another **counter- model** for $A$

By **Decomposition Tree Theorem** all possible restricted counter-models for $A$ are those **determined** by all non-axioms leaves of the $\mathbf{T}_A$

In our case the formula $\mathbf{T}_A$ has only one non-axiom leaf, and hence only one restricted **counter model**

**RS Completeness Theorem**

For any $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash_{\mathbf{RS}} A$$

We prove instead the opposite implication

**RS Completeness Theorem**

$$\text{If} \quad \nvdash_{\mathbf{RS}} A \quad \text{then} \quad \not\models A$$

**Proof** of **Completeness Theorem**

Assume that $A$ is any formula is such that

$$\nvdash_{\mathbf{RS}} A$$

By the **Decomposition Tree Theorem** the $\mathbf{T}_A$ contains a non-axiom leaf

The non-axiom leaf $L_A$ **defines** a truth assignment $v$ which **falsifies** it as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

Hence by **Counter Model Theorem** we have that $v$ also **falsifies** $A$ , i.e.

$$\not\models A$$

PART2:

Proof Systems **RS1** and **RS2**

# RS1 Proof System

Language of **RS1** is the same as the language of **RS**, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

The rules of inference of our system **RS1** operate as rules of **RS** on **finite sequences** of formulas and we adopt

$$\mathcal{E} = \mathcal{F}^*$$

as the set of expressions of **RS1**

**Notation**

Elements of $\mathcal{E}$ are finite sequences of formulas and we denote them by

$$\Gamma, \Delta, \Sigma \ldots$$

with indices if necessary.

# Rules of inference of RS1

Proof System **RS1** contains **seven inference rules**, denoted by the same symbols as the rules of **RS**

$$(\cup), \quad (\neg\cup), \quad (\cap), \quad (\neg\cap), \quad (\Rightarrow), \quad (\neg\Rightarrow), \quad (\neg\neg)$$

The inference rules of **RS1** are quite similar to the rules of **RS**

Look at them **carefully** to see where lies the difference

Reminder

Any propositional variable, or a negation of propositional variable is called a **literal**

The set $LT = VAR \cup \{\neg a : \quad a \in VAR\}$ is called a set of all propositional **literals**

The variables are called **positive literals**

Negations of variables are called **negative literals**.

# Literals Notation

We denote, as before, by

$$\Gamma^{'}, \quad \Delta^{'}, \quad \Sigma^{'} \ldots$$

finite sequences (empty included) formed out of **literals** i.e

$$\Gamma^{'}, \Delta^{'}, \Sigma^{'} \in LT^{*}$$

We will denote by

$$\Gamma, \quad \Delta, \quad \Sigma \ldots$$

the elements of $\mathcal{F}^{*}$

## Logical Axioms of RS1

We adopt all logical axiom of **RS** as the axioms of **RS1**, i.e.

Logical Axioms LA of **RS1** are as follows

$$\Gamma_1^{'},\ a,\ \Gamma_2^{'},\ \neg a,\ \Gamma_3^{'}$$

$$\Gamma_1^{'},\ \neg a,\ \Gamma_2^{'},\ a,\ \Gamma_3^{'}$$

where $a \in VAR$ is any **propositional variable**

## Inference Rules of RS1

**Disjunction rules**

$$(\cup)\ \frac{\Gamma,\ A, B,\ \Delta^{'}}{\Gamma,\ (A \cup B),\ \Delta^{'}} \qquad (\neg\cup)\ \frac{\Gamma,\ \neg A,\ \Delta^{'}\quad ;\quad \Gamma,\ \neg B,\ \Delta^{'}}{\Gamma,\ \neg(A \cup B),\ \Delta^{'}}$$

**Conjunction rules**

$$(\cap)\ \frac{\Gamma,\ A,\ \Delta^{'}\quad ;\quad \Gamma,\ B,\ \Delta^{'}}{\Gamma,\ (A \cap B),\ \Delta^{'}} \qquad (\neg\cap)\ \frac{\Gamma,\ \neg A,\ \neg B,\ \Delta^{'}}{\Gamma,\ \neg(A \cap B),\ \Delta^{'}}$$

# Inference Rules of RS1

## Implication rules

$$(\Rightarrow)\ \frac{\Gamma,\ \neg A, B,\ \Delta^{'}}{\Gamma,\ (A \Rightarrow B),\ \Delta^{'}} \qquad (\neg \Rightarrow)\ \frac{\Gamma,\ A,\ \Delta^{'} \quad : \quad \Gamma,\ \neg B,\ \Delta^{'}}{\Gamma,\ \neg(A \Rightarrow B),\ \Delta^{'}}$$

## Negation rule

$$(\neg\neg)\ \frac{\Gamma,\ A,\ \Delta^{'}}{\Gamma,\ \neg\neg A,\ \Delta^{'}}$$

where $\Gamma^{'} \in LT^{*},\ \ \Delta \in \mathcal{F}^{*},\ \ A, B \in \mathcal{F}$

## Proof System RS1

Formally we define the system **RS1** as follows

$$\textbf{RS1} = (\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}},\ \ \mathcal{E},\ \ LA,\ \ \mathcal{R})$$

where

$$\mathcal{R} = \{(\cup),\ (\neg\cup),\ (\cap),\ (\neg\cap),\ (\Rightarrow),\ (\neg\Rightarrow),\ (\neg\neg)\}$$

for the inference rules is defined above and LA is the set of all logical axioms (the same as for **RS**

## System RS1

**Exercise**

**E1.** Construct a proof in **RS1** of a formula

$$A = (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

**E2.** Prove that **RS1** is **strongly sound**

**E3.** Define in your own words, for any formula $A$, the decomposition tree $\mathbf{T}_A$ in **RS1**

**E4.** Prove **Completeness Theorem** for **RS1**

# System RS1

The decomposition tree $\mathbf{T}_A$ in **RS1** is a **proof** of $A$ in **RS1** as all leaves are axioms

$\mathbf{T}_A$

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\Rightarrow)$$

$$(\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$\mid (\cup)$$

$$\neg\neg(a \cap b), \neg a, \neg b$$

$$\mid (\neg\neg)$$

$$(a \cap b), \neg a, \neg b$$

$$\bigwedge (\cap)$$

$$a, \neg a, \neg b \qquad\qquad b, \neg a, \neg b$$

## Strong Soundness of RS1

**E2.** **Observe** that the system **RS1** is obtained from **RS** by changing the sequence $\Gamma'$ into $\Gamma$ and the sequence $\Delta$ into $\Delta'$ in **all** of the rules of inference of **RS**

These changes do not influence the essence of proof of **strong soundness** of the rules of **RS**

One has just to replace the sequence $\Gamma'$ by $\Gamma$ and $\Delta$ by $\Delta'$ in the the proof of **strong soundness** of each rule of **RS** to obtain the corresponding proof of **strong soundness** of corresponding rule of **RS1**

We do it, for example for the rule $(\cup)$ of **RS1** as follows

## Strong Soundness of RS1

Consider the rule $(\cup)$ of **RS1**

$$(\cup) \quad \frac{\Gamma,\ A, B,\ \Delta^{'}}{\Gamma,\ (A \cup B),\ \Delta^{'}}$$

We evaluate:

$$v^{*}(\Gamma, A, B, \Delta^{'}) = v^{*}(\delta_{\{\Gamma, A, B, \Delta'\}}) = v^{*}(\Gamma) \cup v^{*}(A) \cup v^{*}(B) \cup v^{*}(\Delta^{'})$$

$$= v^{*}(\Gamma) \cup v^{*}(A \cup B) \cup v^{*}(\Delta^{'}) = v^{*}(\delta_{\{\Gamma, (A \cup B), \Delta'\}})$$

$$= v^{*}(\Gamma, (A \cup B), \Delta^{'})$$

# Decomposition Trees in RS1

**E3.** The definition of the decomposition tree $\mathbf{T}_A$ is again, it its essence similar to the one for **RS** except for the changes which reflect the **differences** in the corresponding rules of inference

We follow now the following steps

**Step 1**

Decompose $A$ using a rule defined by its main connective

**Step 2**

Traverse resulting sequence $\Gamma$ on the new node of the tree from **right** to **left** and **find** the first decomposable formula

**Step 3**

Repeat **Step 1** and **Step 2** until no more decomposable formulas

**End of Tree Construction**

## Decomposition Trees in RS1

**E4.** Observe that directly from the definition of the the decomposition tree $\mathbf{T}_A$ we have that the following holds

**Fact 1:** The decomposition tree $\mathbf{T}_A$ is a **proof** iff all leaves are axioms

**Fact 2:** The proof does not exist otherwise, i.e.

$\nvdash_{\mathbf{RS1}} A$ iff there is a non- axiom leaf on $\mathbf{T}_A$

**Fact 2** holds because the tree $\mathbf{T}_A$ is unique

**Observe** that we need **Facts 1, 2** in order to prove **Completeness Theorem** by construction of a **counter-model** generated by a the a non- axiom leaf

**Proof** of **Completeness Theorem**

Assume that $A$ is any formula such that

$$\nvdash_{\mathbf{RS1}} A$$

By **Fact 2** the decomposition tree $\mathbf{T}_A$ contains a non-axiom leaf $L_A$. We use the non-axiom leaf $L_A$ and **define** a truth assignment $v$ which falsifies $A$, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

This proves that

$$\not\models A$$

## System RS2 Definition

**Definition**

System **RS2** is a proof system obtained from **RS** by changing the sequences $\Gamma'$ into $\Gamma$ in **all of the rules** of inference of **RS**

The **logical axioms** LA remind the same

Observe that now the decomposition tree may not be unique

**Exercise 1**

Construct **two** decomposition trees in **RS2** of the formula

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

# RS2 Exercises

**T1**$_A$

$(\neg(\neg a => (a \cap \neg b)) => (\neg a \cap (\neg a \cup \neg b)))$

$\mid (\Rightarrow)$

$\neg\neg(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\Rightarrow)$

$\neg\neg a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a.\neg a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a.\neg a, \neg a, \neg b$

*axiom*

$a, a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a, \neg a, \neg b$

*axiom*

$a, \neg b, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, \neg b, \neg a$

*axiom*

$a, \neg b, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, \neg b, \neg a, \neg b$

*axiom*

# RS2 Exercises

**T2$_A$**

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$$

$$\mid (\neg\neg)$$

$$(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$$

$$\bigwedge (\cap)$$

$$(\neg a \Rightarrow (a \cap \neg b)), \neg a$$

$$\mid (\Rightarrow)$$

$$(\neg\neg a, (a \cap \neg b)), \neg a$$

$$\mid (\neg\neg)$$

$$a, (a \cap \neg b), \neg a$$

$$\bigwedge (\cap)$$

$a, a, \neg a$ *axiom*

$a, \neg b, \neg a$ *axiom*

$$(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cup \neg b)$$

$$\mid (\cup)$$

$$(\neg a \Rightarrow (a \cap \neg b)), \neg a, \neg b$$

$$\mid (\Rightarrow)$$

$$(\neg\neg a, (a \cap \neg b), \neg a, \neg b$$

$$\mid (\neg\neg)$$

$$a, (a \cap \neg b), \neg a, \neg b$$

$$\bigwedge (\cap)$$

$a, a, \neg a, \neg b$ *axiom*

$a, \neg b, \neg a, \neg b$ *axiom*

## System RS2

**Exercise 2** Explain why the system RS2 is **strongly sound**. You can use the Soundness of the system RS

**Solution**

The only one difference between RS and RS2 is that in RS2 each inference rule has at the beginning a sequence of any formulas, not only of literals, as in RS

So there are many ways to **apply rules** as the decomposition rules while constructing the decomposition tree, but it does not affect **strong soundness**, since for all rules of RS2 premisses and conclusions are still logically equivalent as they were in RS

## RS2 Exercises

Consider, for example, **RS2** rule

$$(\cup)\ \frac{\Gamma, A, B, \Delta}{\Gamma, (A \cup B), \Delta}$$

We evaluate

$v^*(\Gamma, A, B, \Delta) = v^*(\Gamma) \cup v^*(A) \cup v^*(B) \cup v^*(\Delta) =$
$v^*(\Gamma) \cup v^*(A \cup B) \cup v^*(\Delta) = v^*(\Gamma, (A \cup B), \Delta)$

Similarly, as in RS, we show all other rules of RS2 to be **strongly sound**, thus RS2 is sound

**Exercise 3**

Define shortly, in your own words, for any formula $A$, its **decomposition tree** $\mathbf{T}_A$ in RS2

**Justify** why your definition is correct

Show that in RS2 the decomposition tree for some formula A may not be unique

**Solution**

Given a formula A. The decomposition tree $\mathbf{T}_A$ can be defined as follows.

It has A as a **root**

For each **node**, if there is a **rule** of RS2 which conclusion has the same form as node sequence, i.e. there is a **decomposition rule** to be applied, then the **node** has children that are **premises** of the **rule**

## RS2 Exercises

If the **node** consists only of literals (i.e. **no decomposition rules** to be applied), then it **does not** have any children

The last statement define a termination condition for the tree

This definition **correctly defines** a decomposition tree for a formula as it identifies and uses appropriate the **decomposition rules**

## RS2 Exercises

Since in RS2 **all rules** of inference have a sequence $\Gamma$ instead of $\Gamma'$ as it was defined for in RS, the **choice** of the decomposition rule for a node may be **not unique**

**For example** consider a **node** $(a \Rightarrow b), (b \cup a)$

$\Gamma$ in the RS2 rules is a sequence of formulas, not literals, so for this **node** we can choose as a **decomposition rule** either $(\Rightarrow)$ or $(\cup)$

This leads to a non-unique tree

## RS2 Exercises

**Exercise 4**

Prove the **Completeness Theorem** for RS2

**Solution**

We need to prove the completeness part only, as the Soundness has been already proved, i.e. we have to prove the implication:

For any formula A , if $\nvdash_{RS2} A$ then $\not\models A$

Assume $\nvdash_{RS2} A$ ,

Then **every** decomposition tree of A has at least one non-axiom leaf

Otherwise, there would **exist** a tree with all axiom leaves and it would be a **proof** for A

# RS2 Exercises

Let $\mathcal{T}_A$ be a set of **all** decomposition trees of $A$

**We choose** an arbitrary $T_A \in \mathcal{T}_A$ with at least one non-axiom leaf $L_A$

The non-axiom leaf $L_A$ **defines** a truth assignment $v$ which falsifies $A$, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

The value for a sequence that corresponds to the leaf in is F

Since, because of the **strong soundness** F "climbs" the tree, we found a **counter-model** for A, i.e.

$$\not\models A$$

**Exercise 5** Write a procedure $TREE_A$ such that for any formula $A$ of **RS2** it produces its **unique** decomposition tree

**Procedure** $TREE_A$(Formula A, Tree T)

{

$B = ChoseLeftMostFormula(A)$ // Choose the left most formula that is not a literal

$c = MainConnective(B)$ // Find the main connective of B

$R = FindRule(c)$// Find the rule which conclusion that has this connective

$P = Premises(R)$// Get the premises for this rule

$AddToTree(A, P)$// add premises as children of A to the tree

For all p in P // go through all premises

$TREE_A(p, T)$ // build subtrees for each premiss

}

**Exercise 6**

Prove **completeness** of your **Procedure** $TREE_A$

**Procedure** $TREE_A$ provides a unique tree, since it always chooses the most left indecomposable formula for a choice of a decomposition rule and there is only one such rule

This procedure is equivalent to RS system, since with thedecomposition rules of RS the most left decomposable formula is always chosen

RS system is **complete**, thus this Procedure is **complete**