cse371/mat371
LOGIC

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Fall 2017

LECTURE 5b

Chapter 5
HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

PART 1: Introduction

PART 2: **Proof** of the **Main Lemma**

PART 3: **Proof 1**: Constructive Proof of **Completeness Theorem**

PART 4: **Proof 2:** General Proof of **Completeness Theorem**

PART 4
Proof 2: General Proof of Completeness Theorem

Proof 2
A Counter- Model Existence Method

We prove now the **Completeness Theorem** by proving the opposite implication:

$$\text{If } \nvdash A, \quad \text{then} \quad \not\models A$$

The **proof** consists of defining a method that uses the information that A is **not provable** in order to define a **counter-model** for A

We hence call it a counter-model existence method.

Proof 2 Steps

The construction of a counter-model for any non-provable A presented in this proof is less constructive then in the case of our first proof.

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

It is hence a much more general method then the first one and this is the reason we present it here.

Proof 2 Steps

We remind that $\not\models A$ means that there is a variable truth assignment $v : VAR \longrightarrow \{T, F\}$, such that as we are in classical semantics $v^*(A) = F$

We assume that A does not have a proof in S, i.e. $\nvdash A$ we use this information in order to define a general method of constructing v, such that $v^*(A) = F$

This is done in the following steps.

Proof 2 Steps

Step 1

Definition of a special set of formulas Δ^*

We use the information $\nvdash A$ to define a set of formulas Δ^* such that $\neg A \in \Delta^*$

Step 2

Definition of the counter - model

We define the variable truth assignment $v : VAR \longrightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if} \quad \Delta^* \vdash a \\ F & \text{if} \quad \Delta^* \vdash \neg a \end{cases}$$

Proof 2 Steps

Step 3

We prove that v is a **counter-model** for A

We first prove a following more general property of v

Property

The set Δ^* and v defined in the Steps 1 and 2 are such that for every formula $B \in \mathcal{F}$

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$

We then use the **Step 3** to prove that $v^*(A) = F$

Main Notions

The definition, construction and the properties of the set Δ^* and hence the **Step 1**, are the most essential for the proof 2

The other steps have mainly technical character

The **main notions** involved in the proof are: consistent set, complete set and a consistent complete extension of a set of formulas

We are going **prove** some essential facts about them.

Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical

Semantical definition uses the notion of a model and says:

A set is **consistent** if it has a **model**

Syntactical definition uses the notion of provability and says:

A set is **consistent** if one can't prove a **contradiction** from it

Consistent and Inconsistent Sets

In our proof of the **Completeness Theorem** we use the following formal syntactical definition of consistency of a set of formulas

Definition of a **consistent set**

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if
there is no a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

Consistent and Inconsistent Sets

Definition of an **inconsistent set**

A set $\Delta \subseteq \mathcal{F}$ is **inconsistent** if and only if **there is** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

The notion of consistency, as defined above, is characterized by the following **Consistency Lemma**

Consistency Condition Lemma

Lemma Consistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) Δ is **consistent**

(ii) **there is** a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$

Proof of Consistency Lemma

Proof

To establish the equivalence of **(i)** and **(ii)** we prove the corresponding opposite implications

We prove the following two cases

Case 1 not **(ii)** implies not **(i)**

Case 2 not **(i)** implies not **(ii)**

Proof of Consistency Lemma

Case 1

Assume that not **(ii)**

It means that **for all formulas** $A \in \mathcal{F}$ we have that

$$\Delta \vdash A$$

In particular it is true for a certain $A = B$ and for a certain $A = \neg B$ i.e.

$$\Delta \vdash B \quad \text{and} \quad \Delta \vdash \neg B$$

and hence it proves that Δ is **inconsistent**

i.e. not **(i)** holds

Proof of Consistency Lemma

Case 2

Assume that not **(i)**, i.e that Δ is **inconsistent**

Then there is a formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$

Let B be any formula

We assumed (**6.**) about S that $\vdash (\neg A \Rightarrow (A \Rightarrow B))$

By monotonicity

$$\Delta \vdash (\neg A \Rightarrow (A \Rightarrow B))$$

Applying Modus Ponens twice to $\neg A$ first, and to A next we get that $\Delta \vdash B$ for any formula B

Thus not **(ii)** and it ends the proof of the **Lemma**

Inconsistency Condition Lemma

Inconsistent sets are hence characterized by the following fact

Lemma Inconsistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **inconsistent**,

(i) for **any formula** $A \in \mathcal{F}$ $\Delta \vdash A$

Finite Consequence Lemma

We remind here property of the finiteness of the **consequence** operation.

Lemma Finite Consequence

For every set Δ of formulas and for every formula $A \in \mathcal{F}$

$\Delta \vdash A$ if and only if there is a **finite** set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$

Proof

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$,
hence by the monotonicity of the consequence, also $\Delta \vdash A$

Finite Consequence Lemma

Assume now that $\Delta \vdash A$ and let

$$A_1, A_2, ..., A_n$$

be a formal proof of A from Δ

Let

$$\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$$

Obviously, Δ_0 is finite and $A_1, A_2, ..., A_n$ is a formal proof of A from Δ_0

Finite Inconsistency Theorem

The following theorem is a simple corollary of just proved Finite Consequence Lemma

Theorem Finite Inconsistency

(1.) If a set Δ is **inconsistent**, then it has a finite **inconsistent** subset Δ_0

(2.) If every finite subset of a set Δ is **consistent** then the set Δ is also **consistent**

Proof

If Δ is **inconsistent**, then for some formula A,

$$\Delta \vdash A \text{ and } \Delta \vdash \neg A$$

By the Finite Consequence Lemma , there are finite subsets Δ_1 and Δ_2 of Δ such that

$$\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A$$

The union $\Delta_1 \cup \Delta_2$ is a finite subset of Δ and by monotonicity

$$\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A$$

Hence we proved that $\Delta_1 \cup \Delta_2$ is a **finite inconsistent subset** of Δ

The second implication **(2)** is the opposite to the one just proved and hence also holds

Consistency Lemma

The following **Lemma** links the notion of non-provability and consistency

It will be used as an important step in our **Proof 2** of the **Completeness Theorem**

Lemma

For any formula $A \in \mathcal{F}$,

if $\nvdash A$ then the set $\{\neg A\}$ is **consistent**

Consistency Lemma

Proof We prove the opposite implication

If $\{\neg A\}$ is **inconsistent**, then $\vdash A$

Assume that $\{\neg A\}$ is **inconsistent**

By the Inconsistency Condition **Lemma** we have that $\{\neg A\} \vdash B$ for **any formula** B, and hence in particular

$$\{\neg A\} \vdash A$$

By **Deduction Theorem** we get

$$\vdash (\neg A \Rightarrow A)$$

We assumed (**9.**) about the system S that

$$((\neg A \Rightarrow A) \Rightarrow A)$$

By Modus Ponens we get

$$\vdash A$$

This **ends the proof**

Complete and Incomplete Sets

Another important notion, is that of a **complete set** of formulas.

Complete sets, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

Definition Complete set

A set Δ of formulas is called **complete** if for every formula $A \in \mathcal{F}$

$$\Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

Godel used this notion of complete sets in his **Incompleteness of Arithmetic Theorem**

The complete sets are characterized by the following fact.

Complete and Incomplete Sets

Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) The set Δ is **complete**

(ii) For every formula $A \in \mathcal{F}$,

if $\Delta \nvdash A$ then then the set $\Delta \cup \{A\}$ is **inconsistent**

Proof

We consider two cases

Case 1 We show that **(i)** implies **(ii)** and

Case 2 we show that **(ii)** implies **(i)**

Complete Set Condition Lemma

Proof of **Case 1**

Assume **(i)** and not**(ii)** i.e.

assume that Δ is **complete** and there is a formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and the set $\Delta \cup \{A\}$ is **consistent**

We have to show that we get a **contradiction**

But if $\Delta \nvdash A$, then from the assumption that Δ is **complete** we get that

$$\Delta \vdash \neg A$$

By the monotonicity of the consequence we have that

$$\Delta \cup \{A\} \vdash \neg A$$

Complete Set Condition Lemma

By assumed provability in S of **4.** $\vdash (A \Rightarrow A)$

By monotonicity $\Delta \vdash (A \Rightarrow A)$ and by **Deduction Theorem**

$$\Delta \cup \{A\} \vdash A$$

We hence proved that that there is a formula $A \in \mathcal{F}$ such that

$$\Delta \cup \{A\} \quad \text{and} \quad \Delta \cup \{A\} \vdash \neg A$$

i.e. that the set $\Delta \cup \{A\}$ is **inconsistent**

Contradiction

Complete Set Condition Lemma

Proof of **Case 2**

Assume **(ii)**, i.e. that for every formula $A \in \mathcal{F}$

if $\Delta \nvdash A$ then the set $\Delta \cup \{A\}$ is **inconsistent**

Let A be any formula.

We want to show **(i)**, i.e. to show that the following condition

$$\mathbf{C}: \quad \Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

is satisfied.

Observe that if

$$\Delta \vdash \neg A$$

then the condition **C** is obviously satisfied

Complete Set Condition Lemma

If, on the other hand,

$$\Delta \nvdash \neg A$$

then we are going to show now that it must be, under the assumption of **(ii)**, that $\Delta \vdash A$ i.e. that **(i)** holds

Assume that

$$\Delta \nvdash \neg A$$

then by **(ii)** the set $\Delta \cup \{\neg A\}$ is **inconsistent**

Complete Set Condition Lemma

The Inconsistency Condition Lemma says

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **inconsistent**,

(i) for any formula $A \in \mathcal{F}$, $\Delta \vdash A$

We just proved that the set $\Delta \cup \{\neg A\}$ is **inconsistent**

So by the the above Lemma we get

$$\Delta \cup \{\neg A\} \vdash A$$

Complete Set Condition Lemma

By the **Deduction Theorem** $\Delta \cup \{\neg A\} \vdash A$ implies that

$$\Delta \vdash (\neg A \Rightarrow A)$$

Observe that

$$((\neg A \Rightarrow A) \Rightarrow A)$$

is a provable formula **4.** in S

By monotonicity

$$\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A)$$

Detaching, by MP the formula $(\neg A \Rightarrow A)$ we obtain that

$$\Delta \vdash A$$

This **ends** the proof that **(i)** holds.

Incomplete Sets

Definition Incomplete Set

A set Δ of formulas is called **incomplete** if it is **not complete** i.e. when the following condition holds

There exists a formula $A \in \mathcal{F}$ such that

$$\Delta \nvdash A \quad \text{and} \quad \Delta \nvdash \neg A$$

Incomplete Set Condition Lemma

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of **incomplete sets**

Lemma Incomplete Set Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **incomplete**,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \nvdash A$ and the set $\Delta \cup \{A\}$ is **consistent**.

Main Lemma: Complete Consistent Extension

Now we are going to prove a **Lemma** that is essential to the construction of the special set Δ^* mentioned in the **Step 1** of the proof of the **Completeness Theorem** and hence to the **proof of the theorem** itself

Let's first introduce one more notion

Complete Consistent Extension

Definition Extension Δ^* of the set Δ

A set Δ^* of formulas is called an **extension** of a set Δ of formulas if the following **condition holds**

$$\{A \in \mathcal{F} : \ \Delta \vdash A\} \subseteq \{A \in \mathcal{F} : \ \Delta^* \vdash A\}$$

i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

In this case **we say** also that Δ **extends** to the set of formulas Δ^*

Complete Consistent Extension

The Main Lemma Complete Consistent Extension

Every **consistent** set Δ of formulas can be **extended** to a **complete consistent** set Δ^* of formulas
i. e

For every **consistent** set Δ there is a set Δ^* that is **complete** and **consistent** and is an **extension** of Δ i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

Proof of the Main Lemma

Proof

Assume that the lemma does not hold, i.e. that there is a **consistent** set Δ, such that all its consistent extensions are **not complete**

In particular, as Δ is an consistent extension of itself, we have that Δ is **not complete**

The proof consists of a **construction** of a particular set Δ^* and **proving** that it forms a **complete** consistent extension of Δ

This is **contrary** to the assumption that all its consistent extensions are **not complete**

Construction of Δ^*

As we know, the set $\mathcal{F}$ of all formulas is enumerable; they can hence be put in an infinite sequence

$$\textbf{F} \qquad A_1, A_2, \ \ldots \ , A_n, \ \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once

We define, by **mathematical induction**, an infinite sequence

$$\textbf{D} \qquad \{\Delta_n\}_{n \in N}$$

of **consistent** subsets of formulas together with a sequence

$$\textbf{B} \qquad \{B_n\}_{n \in N}$$

of formulas as follows

Construction of Δ^*

Initial Step

In this step we define the sets

$$\Delta_1, \Delta_2 \quad \text{and the formula} \quad B_1$$

and **prove** that

$$\Delta_1 \quad \text{and} \quad \Delta_2$$

are **consistent, incomplete** extensions of Δ

We take as the first set in **D** the set Δ, i.e. we define

$$\Delta_1 = \Delta$$

Construction of Δ^*

By assumption the set Δ, and hence also Δ_1 is **not complete**.

From the Incomplete Set Condition **Lemma** we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \nvdash B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is} \ \textbf{consistent}$$

Let B_1 be the **first formula** with this property in the sequence **F** of all formulas

We **define**

$$\Delta_2 = \Delta_1 \cup \{B_1\}$$

Construction of Δ^*

Observe that the set Δ_2 is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2$$

By monotonicity Δ_2 is a **consistent extension** of Δ

Hence, as we assumed that all consistent extensions of Δ are **not complete**, we get that Δ_2 cannot be complete, i.e.

Δ_2 is **incomplete**

Construction of Δ^*

Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \Delta_2, \ldots, \Delta_n$$

of **incomplete, consistent extensions** of Δ and a sequence

$$B_1, B_2, \ldots, B_{n-1}$$

of formulas, for $n \geq 2$

Construction of Δ^*

Since Δ_n is **incomplete**, it follows from the Incomplete Set Condition **Lemma** that

there is a formula $B \in \mathcal{F}$ such that

$$\Delta_n \nvdash B \quad \text{and} \quad \Delta_n \cup \{B\} \quad \text{is } \textbf{consistent}$$

Construction of Δ^*

Let B_n be the first formula with this property in the sequence **F** of all formulas.

We **define**

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}$$

By the definition

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set Δ_{n+1} is a **consistent** extension of Δ

Hence by our assumption that all all consistent extensions o f Δ are **incomplete** we get that

$$\Delta_{n+1}$$

is an **incomplete** consistent extension of Δ

Construction of Δ^*

By the principle of mathematical induction we have defined an infinite sequence

$$\mathbf{D} \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq ....$$

such that for all $n \in N$, Δ_n is **consistent**, and each Δ_n an **incomplete consistent extension** of Δ

Moreover, we have also defined a sequence

$$\mathbf{B} \qquad B_1, B_2, \ldots, B_n, \ldots$$

of formulas, such that for all $n \in N$,

$$\Delta_n \nvdash B_n \quad \text{and} \quad \Delta_n \cup \{B_n\} \quad \text{is } \textbf{consistent}$$

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$

Definition of Δ^*

Now we are ready to define Δ^*

Definition of Δ^*

$$\Delta^* = \bigcup_{n \in N} \Delta_n$$

To complete the proof our theorem we have now to prove that Δ^* is a **complete consistent extension** of Δ

Δ^* Consistent

Obviously directly from the definition $\Delta \subseteq \Delta^*$ and hence we have the following

Fact 1 Δ^* is an **extension** of Δ

By Monotonicity of Consequence $Cn(\Delta) \subseteq Cn(\Delta^*)$, hence extension

As the next step we prove

Fact 2 The set Δ^* is **consistent**

Δ^* Consistent

Proof that Δ^* is **consistent**

Assume that Δ^* is **inconsistent**

By the Finite Inconsistency Theorem there is a finite subset Δ_0 of Δ^* that is **inconsistent**, i.e.

$\Delta_0 \subseteq \bigcup_{n \in N} \Delta_n, \quad \Delta_0 = \{C_1, ..., C_n\}, \quad \Delta_0$ is **inconsistent**

Proof of Δ^* Consistent

We have $\Delta_0 = \{C_1, \ldots, C_n\}$

By the definition of Δ^* for each formula $C_i \in \Delta_0$

$$C_i \in \Delta_{k_i}$$

for certain Δ_{k_i} in the sequence

$$\mathbf{D} \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq ....$$

Hence $\Delta_0 \subseteq \Delta_m$ for $m = max\{k_1, k_2, ..k_n\}$

Proof of Δ^* Consistent

But we proved that all sets of the sequence **D** are **consistent**

This contradicts the fact that Δ_m is **consistent** as it contains an **inconsistent** subset Δ_0

This contradiction ends the proof that Δ^* is **consistent**

Proof of Δ^* Complete

Fact 3 The set Δ^* is **complete**

Proof Assume that Δ^* is **not complete**.

By the Incomplete Set Condition, there is a formula $B \in \mathcal{F}$ such that

$\Delta^* \nvdash B$, and the set $\Delta^* \cup \{B\}$ is **consistent**

By definition of the sequence **D** and the sequence **B** of formulas we have that for every $n \in N$

$$\Delta_n \nvdash B_n \quad \text{and the set} \quad \Delta_n \cup \{B_n\} \ \text{is} \ \textbf{consistent}$$

Moreover $B_n \in \Delta_{n+1}$ for all $n \geq 1$

Proof of Δ^* Complete

Since the formula B is one of the formulas of the sequence **B** so we get that $B = B_j$ for certain j

By definition, $B_j \in \Delta_{j+1}$ and it proves that

$$B \in \Delta^* = \bigcup_{n \in N} \Delta_n$$

But this means that $\Delta^* \vdash B$

This is a contradiction with the assumption $\Delta^* \nvdash B$ and it **ends the proof** of the **Fact 3**

Facts 1- 3 prove that that Δ^* is a **complete consistent extension** of Δ and completes the proof out **Main Lemma**

Proof 2 of Completeness Theorem

Proof 2 of Completeness Theorem

As by assumption our proof system S is sound, we have to prove only the Completeness part of the **Completeness Theorem**, i.e to prove that

Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash A$$

We prove it by proving the opposite implication

Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\text{If} \quad \nvdash A, \quad \text{then} \quad \not\models A$$

Proof 2 of Completeness Theorem

Proof

Assume that A **doesn't have a proof** in S, we want to define a **counter-model** for A

But if $\nvdash A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is **consistent**

By the **Main Lemma** there is a complete, consistent extension of the set $\{\neg A\}$

This means that there is a set Δ^* such that $\{\neg A\} \subseteq \Delta^*$, i.e.

E $\quad \neg A \in \Delta^*$ and Δ^* is **complete** and **consistent**

Proof 2 of Completeness Theorem

Since Δ^* is a **consistent, complete** set, it satisfies the following form of

Consistency Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \nvdash A \quad \text{or} \quad \Delta^* \nvdash \neg A$$

Δ^* i s also **complete** i.e. satisfies

Completeness Condition

For any $A \in \mathcal{F}$,

$$\Delta^* \vdash A \quad \text{or} \quad \Delta^* \vdash \neg A$$

Proof 2 of Completeness Theorem

Directly from the Completeness and Consistency Conditions we get the following

Separation Condition

For any $A \in \mathcal{F}$, **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or **(2)** $\Delta^* \vdash \neg A$

In particular case we have that for every propositional variable $a \in VAR$ **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or **(2)** $\Delta^* \vdash \neg a$

This justifies the **correctness** of the following definition

Proof 2 of Completeness Theorem

Definition

We define the variable truth assignment

$$v : VAR \longrightarrow \{T, F\}$$

as follows:

$$v(a) = \begin{cases} T & \text{if } \Delta^* \vdash a \\ F & \text{if } \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate Lemma below, that such defined variable assignment v has the following property

Property of v Lemma

Lemma Property of v

Let v be the variable assignment defined above and v^* its extension to the set $\mathcal{F}$ of all formulas $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$

Proof 2 of Completeness Theorem

Given the Property of v Lemma (still to be proved)

we now **prove** that the v is in fact, a **counter model** for any formula A, such that $\nvdash A$

Let A be such that $\nvdash A$

By the Property **E** we have that $\neg A \in \Delta^*$

So obviously

$$\Delta^* \vdash \neg A$$

Hence by the Property of v Lemma

$$v^*(A) = F$$

what **proves** that v is a **counter-model** for A and it **ends the proof** of the Completeness Theorem

Proof of Property of v Lemma

Proof of the Property of v **Lemma**

The proof is conducted by the induction on the degree of the formula A

Initial step A is a propositional variable so the **Lemma** holds by definition of v

Inductive Step

If A is not a propositional variable, then A is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas C, D

By the inductive assumption the **Lemma** holds for the formulas C and D

Proof of Property of v Lemma

Case $A = \neg C$

By the **Separation Condition** for Δ^* we consider two possibilities

1. $\Delta^* \vdash A$

2. $\Delta^* \vdash \neg A$

Consider case **1.** i.e. we assume that $\Delta^* \vdash A$

It means that

$$\Delta^* \vdash \neg C$$

Then from the fact that Δ^* is **consistent** it must be that

$$\Delta^* \nvdash C$$

Proof of Property of v Lemma

By the inductive assumption we have that $v^*(C) = F$ and accordingly $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$

Consider case **2.** i.e. we assume that $\Delta^* \vdash \neg A$

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \nvdash A$ and

$$\Delta^* \nvdash \neg C$$

If so, then $\Delta^* \vdash C$, as the set Δ^* is **complete**

By the **inductive assumption**, $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F$$

Thus A **satisfies** the Property of v Lemma.

Proof of Property of v Lemma

Case $A = (C \Rightarrow D)$

As in the previous case, we assume that the Lemma holds for the formulas C, D and we consider by the **Separation Condition** for Δ^* two possibilities:

1. $\Delta^* \vdash A$ and **2.** $\Delta^* \vdash \neg A$

Case 1. Assume $\Delta^* \vdash A$

It means that $\Delta^* \vdash (C \Rightarrow D)$

If at the same time $\Delta^* \nvdash C$, then $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = T$$

Proof of Property of v Lemma

If at the same time $\Delta^* \vdash C$, then since $\Delta^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that

$$\Delta^* \vdash D$$

If so, then $v^*(C) = v^*(D) = T$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T$$

Thus if $\Delta^* \vdash A$, then $v^*(A) = T$

Proof of Property of v Lemma

Case 2. Assume now, as before, that $\Delta^* \vdash \neg A$,

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \nvdash A$, i.e.,

$$\Delta^* \nvdash (C \Rightarrow D)$$

It follows from this that $\Delta^* \nvdash D$

For if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula **1.** in S, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we obtain

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption, so it must be $\Delta^* \nvdash D$

Proof of Property of v Lemma

Also we must have

$$\Delta^* \vdash C$$

for otherwise, as Δ^* is **complete** we would have

$$\Delta^* \vdash \neg C$$

But this is **impossible** since the formula $(\neg C \Rightarrow (C \Rightarrow D))$ is assumed to be provable formula **9.** in S and by monotonicity

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we would get

$$\Delta^* \vdash (C \Rightarrow D)$$

which is **contrary** to the assumption $\Delta^* \nvdash (C \Rightarrow D)$

This **ends the proof** of the Property of v Lemma and hence the proof of the Completeness Theorem is also **completed**