cse371/mat371
LOGIC

Professor Anita Wasilewska
LECTURE 3d
Chapter 3 REVIEW (1)
Some Definitions and Problems
DEFINITIONS: Part One

There are some basic DEFINITIONS from Chapter 3

You have to prepare them for Quiz

I will ask you to WRITE down a full, correct text of 1-3 of them - in EXACTLY the same form as they are presented here

Knowing all basic Definitions is the first step to understanding the material
DEFINITIONS: Propositional Extensional Semantics

Definition 1
Given a propositional language \( L_{\text{CON}} \) for the set \( \text{CON} = C_1 \cup C_2 \), where \( C_1, C_2 \) are respectively the sets of unary and binary connectives
Let \( V \) be a non-empty set of logical values
Connectives \( \triangledown \in C_1 \), \( \circ \in C_2 \) are called extensional iff their semantics is defined by respective functions

\[ \triangledown : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V \]
DEFINITIONS: Propositional Extensional Semantics

Definition 2
Formal definition of a **propositional extensional semantics** for a given language $\mathcal{L}_{\text{CON}}$ consists of providing **definitions** of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology
CLASSICAL PROPOSITIONAL SEMANTICS
DEFINITIONS: Truth Assignment Extension $v^*$

Definition 3

The Language: $L = L_{\neg,\Rightarrow,\cup,\cap}$

Given the truth assignment $v : VAR \rightarrow \{ T, F \}$ in classical semantics for the language $L = L_{\neg,\Rightarrow,\cup,\cap}$, we define its extension $v^*$ to the set $F$ of all formulas of $L$ as $v^* : F \rightarrow \{ T, F \}$ such that

(i) for any $a \in VAR$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in F$ we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$
$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$
$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$
$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B)).$$
DEFINITIONS: Truth Assignment Extension $v^*$ Revisited

Notation
For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in case arithmetic operations.

The condition (ii) of the definition of the extension $v^*$ can be hence written as follows:

(ii) and for any $A, B \in F$ we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$
$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$
$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$
$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B).$$
DEFINITIONS: Satisfaction Relation

Definition 4  Let \( v : \text{VAR} \rightarrow \{T, F\} \)
We say that \( v \) satisfies a formula \( A \in \mathcal{F} \) iff \( v^*(A) = T \)

Notation: \( v \models A \)
We say that \( v \) does not satisfy a formula \( A \in \mathcal{F} \) iff \( v^*(A) \neq T \)

Notation: \( v \not\models A \)
DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5
Given a formula \( A \in \mathcal{F} \) and \( v : \text{VAR} \rightarrow \{T, F\} \)
We say that
\( v \) is a model for \( A \) iff \( v \models A \)
\( v \) is a counter-model for \( A \) iff \( v \not\models A \)

Definition 6
\( A \) is a tautology iff for any \( v : \text{VAR} \rightarrow \{T, F\} \) we have that \( v \models A \)

Notation
We write symbolically \( \models A \) to denote that \( A \) is a classical tautology
DEFINITIONS: Restricted Truth Assignments

**Notation:** for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

**Definition 7** Given a formula $A \in \mathcal{F}$, any function $v_A : \text{VAR}_A \longrightarrow \{T, F\}$

is called a truth assignment restricted to $A$
DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$.

Definition 8: Given a formula $A \in \mathcal{F}$.
Any function

$$w : \ \VAR_A \longrightarrow \{T, F\} \ \text{such that} \ w^*(A) = T$$

is called a **restricted MODEL** for $A$.

Any function

$$w : \ \VAR_A \longrightarrow \{T, F\} \ \text{such that} \ w^*(A) \neq T$$

is called a **restricted Counter- MODEL** for $A$.
Consider \( \mathcal{L} = \mathcal{L}\{\neg, \cup, \cap, \Rightarrow\} \) and let \( S \neq \emptyset \) be any non empty set of formulas of \( \mathcal{L} \), i.e.

\[
S \subseteq F
\]

**Definition 9**

A truth assignment \( \nu : \text{VAR} \rightarrow \{T, F\} \) is a **model for the set** \( S \) of formulas if and only if

\[
\nu \models A \quad \text{for all formulas} \quad A \in S
\]

We write

\[
\nu \models S
\]

to denote that \( \nu \) is a **model for the set** \( S \) of formulas.
DEFINITIONS: Consistent Sets of Formulas

Definition 10
A non-empty set \( G \subseteq F \) of formulas is called consistent if and only if \( G \) has a model, i.e. we have that

\[
G \subseteq F \quad \text{is consistent} \quad \text{if and only if}
\]

there is \( v \) such that \( v \models G \)

Otherwise \( G \) is called inconsistent.
DEFINITIONS: Independent Statements

Definition 11
A formula $A$ is called independent from a non-empty set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is independent if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\} \quad \text{are consistent}$$
Many Valued Extensional Semantics
Definition 11
The extensional semantics $M$ is defined for a non-empty set of $V$ of logical values of any cardinality.

We only assume that the set $V$ of logical values of $M$ always has a special, distinguished logical value which serves to define a notion of tautology.

We denote this distinguished value as $T$.

Formal definition of many valued extensional semantics $M$ for the language $L_{CON}$ consists of giving definitions of the following main components:

1. Logical Connectives under semantics $M$
2. Truth Assignment for $M$
3. Satisfaction Relation, Model, Counter-Model under semantics $M$
4. Tautology under semantics $M$
Definition of $\mathbf{M}$ - Extensional Connectives

Given a propositional language $\mathcal{L}_{\text{CON}}$ for the set $\text{CON} = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives

Let $V$ be a non-empty set of logical values adopted by the semantics $\mathbf{M}$

**Definition 12**

Connectives $\bigtriangledown \in C_1$, $\circ \in C_2$ are called $\mathbf{M}$-extensional iff their semantics $\mathbf{M}$ is defined by respective functions

$$\bigtriangledown : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITION: Definability of Connectives under a semantics $M$

Given a propositional language $L_{CON}$ and its extensional semantics $M$

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ **under the semantics $M$** if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are defined by the semantics $M$
DEFINITION: M Truth Assignment Extension $v^*$ to $\mathcal{F}$

Definition 14
Given the M truth assignment $v : \text{VAR} \rightarrow V$
We define its M extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function $v^* : \mathcal{F} \rightarrow V$, such that the following conditions are satisfied

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$
DEFINITION: M Satisfaction, Model, Counter Model, Tautology

Definition 15  Let \( v : \text{VAR} \rightarrow V \)
Let \( T \in V \) be the distinguished logical value
We say that
\( v \) M satisfies a formula \( A \in F \) (\( v \models_M A \)) iff \( v^*(A) = T \)

Definition 16
Given a formula \( A \in F \) and \( v : \text{VAR} \rightarrow V \)
Any \( v \) such that \( v \models_M A \) is called a M model for \( A \)
Any \( v \) such that \( v \not\models_M A \) is called a M counter model for \( A \)
\( A \) is a M tautology (\( \models_M A \)) iff \( v \models_M A \), for all \( v : \text{VAR} \rightarrow V \)
CHAPTER 3: Some Questions
Chapter 3: Question 1

Question 1

1. Find a restricted model for formula $A$, where

$$A = (\neg a \implies (\neg b \cup (b \implies \neg c)))$$

You can’t use short-hand notation
Show each step of solution

Solution

For any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$
Chapter 3: Question 1

Let $v : VAR \rightarrow \{T, F\}$ be any truth assignment such that

$$v(a) = v_A(a) = T, \ v(b) = v_A(b) = T, \ v(c) = v_A(c) = F$$

We evaluate the value of the extension $v^*$ of $v$ on the formula $A$ as follows

$$v^*(A) = v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))$$
$$= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c)))$$
$$= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c)))$$
$$= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c)))$$
$$= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)))$$

$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$, i.e.

$$v_A \models A \quad \text{and} \quad v \models A$$
Chapter 3: Question 2

Question 2
1. Find a restricted model and a restricted counter-model for $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can use short-hand notation. Show work

Solution

Notation: for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$

In our case we have $VAR_A = \{a, b, c\}$

Any function $v_A : VAR_A \longrightarrow \{T, F\}$ is called a truth assignment restricted to $A$

We define now $v_A(a) = T$, $v_A(b) = T$, $v_A(c) = F$, in shorthand: $a = T$, $b = T$, $c = F$ and evaluate

$$(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T$$

i.e.

$$v_A \models A$$
Observe that
\[(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T \quad \text{when } a = T \quad \text{and } b, c \text{ any truth values as by definition of implication we have that } F \Rightarrow \text{anything} = T\]
Hence \(a = T\) gives us 4 models as we have \(2^2\) possible values on \(b\) and \(c\).
Chapter 3: Question 2

We take as a restricted counter-model: \(a = F\), \(b = T\) and \(c = T\)

**Evaluation:** observe that

\[\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = F\] if and only if

\[\neg a = T\] and \((\neg b \cup (b \Rightarrow \neg c)) = F\) if and only if

\(a = F\), \(\neg b = F\) and \((b \Rightarrow \neg c) = F\) if and only if

\(a = F\), \(b = T\) and \((T \Rightarrow \neg c) = F\) if and only if

\(a = F\), \(b = T\) and \(\neg c = F\) if and only if

\(a = F\), \(b = T\) and \(c = T\)

The above proves also that \(a = F\), \(b = T\) and \(c = T\) is the only restricted counter-model for \(A\)
Chapter 3: Question 3

Question 3  Justify whether the following statements true or false

S1  There are more then 3 possible restricted counter-models for $A$

S2  There are more then 2 possible restricted models of $A$

Solution

Statement:  There are more then 3 possible restricted counter-models for $A$ is false

We have just proved that there is only one possible restricted counter-model for $A$

Statement:  There are more then 2 possible restricted models of $A$ is true

There are 7 possible restricted models for $A$

Justification:  $2^3 - 1 = 7$
Chapter 3: Question 4

Question 4
1. List 3 models and 2 counter-models for A from Question 2, i.e. for formula

\[ A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \]

that are extensions to the set VAR of all variables of one the restricted models and of one of the restricted counter-models that you have found in Questions 1,2
Chapter 3: Question 4

Solution
One of the restricted models is, for example a function
\[ v_A : \{a, b, c\} \rightarrow \{T, F\} \] such that
\[ v_A(a) = T, \ v_A(b) = T, \ v_A(c) = F \]

We extend \( v_A \) to the set of all propositional variables \( VAR \) to obtain a (non restricted) models as follows
Chapter 3: Question 4

Model $w_1$ is a function

$$w_1 : VAR \rightarrow \{T, F\}$$ such that

- $w_1(a) = v_A(a) = T$, $w_1(b) = v_A(b) = T$,
- $w_1(c) = v_A(c) = F$, and $w_1(x) = T$, for all $x \in VAR - \{a, b, c\}$

Model $w_2$ is defined by a formula

- $w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,
- $w_2(c) = v_A(c) = F$, and $w_2(x) = F$, for all $x \in VAR - \{a, b, c\}$
Chapter 3: Question 4

Model \( w_3 \) is defined by a formula
\[ w_3(a) = v_A(a) = T, \quad w_3(b) = v_A(b) = T, \quad w_3(c) = v(c) = F, \]
\[ w_3(d) = F \quad \text{and} \quad w_3(x) = T \quad \text{for all} \quad x \in VAR - \{a, b, c, d\} \]

There is as many of such models, as extensions of \( v_A \) to the set \( VAR \), i.e. as many as real numbers
Chapter 3: Question 4

A counter-model for a formula $A$, by definition, is any function

$$v : \text{VAR} \rightarrow \{T, F\}$$

such that $v^*(A) = F$

A restricted counter-model for $A$ (only one as proved in question 5) is a function

$$v_A : \{a, b\} \rightarrow \{T, F\}$$

such that

$$v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T$$
Chapter 3: Question 4

We extend $v_A$ to the set of all propositional variables $VAR$ to obtain (non restricted ) some counter-models.

Here are two of such extensions

**Counter-model $w_1$:**

$w_1(a) = v_A(a) = F$, $w_1(b) = v_A(b) = T$,
$w_1(c) = v(c) = T$, and $w_1(x) = F$, for all $x \in VAR \setminus \{a, b, c\}$

**Counter-model $w_2$:**

$w_2(a) = v_A(a) = T$, $w_2(b) = v_A(b) = T$,
$w_2(c) = v(c) = T$, and $w_2(x) = T$ for all $x \in VAR \setminus \{a, b, c\}$

There is as many of such counter-models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers
Definition

A truth assignment \( v \) is a **model for a set** \( G \subseteq F \) of **formulas** of a given language \( L = L\{\neg, \Rightarrow, \cup, \cap\} \) if and only if

\[ v \models B \quad \text{for all} \quad B \in G \]

We denote it by \( v \models G \)

Observe that the set \( G \subseteq F \) can be **finite** or **infinite**
Chapter 3: Consistent Sets of Formulas

Definition
A set $G \subseteq \mathcal{F}$ of formulas is called consistent if and only if $G$ has a model, i.e. we have that $G \subseteq \mathcal{F}$ is consistent if and only if there is $v$ such that $v \models G$

Otherwise $G$ is called inconsistent
Chapter 3: Independent Statements

Definition
A formula $A$ is called independent from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$v_1 \models G \cup \{A\}$ and $v_2 \models G \cup \{\neg A\}$

i.e. we say that a formula $A$ is independent if and only if

$G \cup \{A\}$ and $G \cup \{\neg A\}$ are consistent
Question 5

Given a set

\[ G = \{ ((a \cap b) \Rightarrow b), (a \cup b), \neg a \} \]

Show that \( G \) is consistent

Solution

We have to find \( v : \text{VAR} \rightarrow \{T, F\} \) such that

\[ v \models G \]

It means that we need to find \( v \) such that

\[ v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T \]
Chapter 3: Question 5

Observe that $\models ((a \cap b) \Rightarrow b)$, hence we have that

1. $v^*((a \cap b) \Rightarrow b) = T$ for any $v$

   $v^*(\neg a) = \neg v^*(a) = \neg v(a) = T$ only when $v(a) = F$ hence

2. $v(a) = F$

   $v^*(a \cup b) = v^*(a) \cup v^*(b) = v(a) \cup v(b) = F \cup v(b) = T$

   only when $v(b) = T$ so we get

3. $v(b) = T$

   This means that for any $v : VAR \rightarrow \{ T, F \}$ such that $v(a) = F, \ v(b) = T$

   $v \models G$

   and we proved that $G$ is consistent
Chapter 3: Question 6

Question 6
Show that a formula \( A = (\neg a \cap b) \) is not independent of \( G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \)

Solution
We have to show that it is impossible to construct \( v_1, v_2 \) such that

\[
v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}
\]

Observe that we have just proved that any \( v \) such that \( v(a) = F, \) and \( v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) for \( G \) so we have to check now if it is possible that \( v \models A \) and \( v \models \neg A \)
We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for $v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \land b)) = \neg v(a) \land v(b) = \neg F \land T = T \land T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This ends the proof that $A$ is not independent of $G$
Chapter 3: Question 7

Question 7

2. Find an infinite number of formulas that are independent of \( G = \{((a \land b) \Rightarrow b), (a \lor b), \neg a\} \)

This my solution - there are many others- this one seemed to me the most simple

Solution

We just proved that any \( v \) such that \( v(a) = F, v(b) = T \) is the only model restricted to the set of variables \{a, b\} and so all other possible models for \( G \) must be extensions of \( v \)
Chapter 3: Question 7

We define a countably infinite set of formulas (and their negations) and corresponding extensions of $v$ (restricted to to the set of variables $\{a, b\}$) such that $v \models G$ as follows.

Observe that all extensions of $v$ restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$\text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

$$\mathcal{F}_0 = \text{VAR} - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$
Let $c \in \mathcal{F}_0$
We define truth assignments $\nu_1, \nu_2 : \text{VAR} \rightarrow \{T, F\}$ such that

\[\nu_1 \models G \cup \{c\} \quad \text{and} \quad \nu_2 \models G \cup \{\neg c\}\]

as follows

\[\nu_1(a) = \nu(a) = F, \quad \nu_1(b) = \nu(b) = T \quad \text{and} \quad \nu_1(c) = T\]
for all $c \in \mathcal{F}_0$

\[\nu_2(a) = \nu(a) = F, \quad \nu_2(b) = \nu(b) = T \quad \text{and} \quad \nu_2(c) = F\]
for all $c \in \mathcal{F}_0$
CHAPTER 3
Some Extensional Many Valued Semantics
Chapter 3: Question 8

Question 8
We define a 4 valued $H_4$ logic semantics as follows.

The language is $L = \mathcal{L}\{\neg, \Rightarrow, \cup, \cap\}$

The logical connectives $\neg$, $\Rightarrow$, $\cup$, $\cap$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows.

Conjunction $\cap$ is a function

$\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$

$$a \cap b = \min\{a, b\}$$
Chapter 3: Many Valued Semantics

**Disjunction** $\cup$ is a function

$\cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$

$$a \cup b = \max\{a, b\}$$

**Implication** $\Rightarrow$ is a function

$\Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$, such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

**Negation:**

$$\neg a = a \Rightarrow F$$
Chapter 3: Question 10

Part 1  Write Truth Tables for IMPLICATION and NEGATION in $H_4$

Solution

$H_4$ Implication

<table>
<thead>
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<th>$\Rightarrow$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_1$</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_2$</td>
<td>F</td>
<td>$\bot_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\bot_1$</td>
<td>$\bot_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Chapter 3: Question 10

Part 2   Verify whether

\[ \models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]

Solution

Take any \( v \) such that

\( v(a) = \bot_1 \quad v(b) = \bot_2 \)

Evaluate

\( v \ast ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2)) = T \Rightarrow \bot_2 = \bot_2 \)

This proves that our \( v \) is a counter-model and hence

\[ \not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Chapter 3: Classical Propositional Tautologies

Question 11
Show that (can’t use TTables!)

\[ \vdash ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)
Our formula becomes a substitution of a basic tautology

\[ (A \Rightarrow (B \Rightarrow A)) \]

and hence is a tautology
Chapter 3: Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$ - it means write carefully all Steps 1- 4 of the definition of your $M$