cse371/mat371
LOGIC

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LECTURE 3b

CHAPTER 3
Some Extensional Many Valued Semantics

CHAPTER 5

PART 1: Some Three Valued Extensional Semantics

PART 2: Many Valued Extensional Semantics **M**

CHAPTER 5

PART 1: Some Three Valued Extensional Semantics

First Many Valued Logics

First many valued logic (defined semantically only) was formulated by Łukasiewicz in 1920

We present here some of the first 3-valued extensional semantics, historically called 3-valued logics

They are named after their authors: Łukasiewicz, Kleene, Heyting, and Bochvar

We assume that the **language** of all logics logics considered except of Bochvar logic is

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}}$$

3-Valued Semantics

We add one extra logical value $\perp$ to the classical set of two values $\{T, F\}$ to be able to express that the logical value of a statement A may now be **not only** true or false

The **third logical value** denotes a notion of "unknown", "uncertain", "undefined", or even can express that "we don't have a complete information about A", depending on the context and **motivation** for the logic

$\perp$ is the most frequently used **symbol** for different concepts of **unknown**

3 Valued Semantics Assumptions

All three valued logics considered here, when defined semantically, enlist a third logical value which we denote by $\perp$, or m in case of Bochvar semantics

We assume that the third value is **intermediate** between truth and falsity, i.e. the set of logical values is **ordered** and we have the following

Assumption 1

$$F < \perp < T, \quad \text{and} \quad F < m < T$$

Assumption 2

In all of presented here semantics we take T as **designated value**, i.e. T is the value that **defines** the notion of **satisfiability** and **tautology**

Many Valued Semantics Assumptions

The third value $\perp$ corresponds also to some notion of incomplete information, inconsistent information, or to a notion of being undefined , or unknown

Historically all these semantics, and many others, were and still are called logics

We also will use the name logic for them, instead saying each time "logic defined semantically", or "semantics for a given logic"

Many Valued Extensional Semantics

Reminder: we assumed that in all cases, except of Bochvar logic the language is

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}}$$

Formal definition of **many valued extensional semantics** follows the Definition of Extensional Semantics in general and the pattern of presented in detail for the classical case in particular

It consists of giving **definitions** of the following main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction Relation, Model, Counter-Model
4. Tautology

We define **all the steps** in case of Łukasiewicz' s semantics (logic) to establish a pattern and proper notation and leave in case of other logics as an exercise for the reader

Łukasiewicz Logic Ł

Motivation

Łukasiewicz developed his semantics (called logic) to deal with future contingent statements

Contingent statements are not just neither true nor false but are indeterminate in some metaphysical sense

It is not only that we do not know their truth value but rather that they do not possess one

Ł Language

The Language:

$$\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

Observe that the language is the same as in the **classical** case

The set $\mathcal{F}$ of **formulas** is defined in a standard way

Ł Semantics: Connectives

Step 1 of Ł semantics definition

Remember that we assumed: $F < \perp < T$

Ł Negation $\neg$ is a **function:**

$$\neg : \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that $\neg \perp = \perp, \quad \neg T = F, \quad \neg F = T$

Ł Conjunction $\cap$ is a **function:**

$$\cap : \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put $x \cap y = min\{x, \ y\}$

Ł Semantics: Connectives

Remember that we assumed: $F < \perp < T$

Ł Disjunction $\cup$ is a **function:**

$$\cup : \ \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(a, b) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put
$x \cup y = max\{x, \ y\}$

Ł Implication $\Rightarrow$ is a **function:**

$$\Rightarrow : \ \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$, we put

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

Ł Connectives Truth Tables

Ł Negation

$\neg$	F	$\perp$	T
	T	$\perp$	F

Ł Conjunction

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

Ł Connectives Truth Tables

Ł Disjunction

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

Ł Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

Ł - Semantics: Truth Assignment

Step 2 of Ł -semantics **definition**

Definition

A **truth assignment** is now **any function**

$$v : VAR \longrightarrow \{F, \perp, T\}$$

Observe that the domain of **truth assignment** is the set of propositional variables, i.e. the **truth assignment** is defined only for **atomic formulas**

Truth Assignment Extension v^* to $\mathcal{F}$

Definition

Given a truth assignment $v : VAR \longrightarrow \{T, \perp, F\}$

We define its **extension** $v^* : \mathcal{F} \longrightarrow \{T, \perp, F\}$ by the **induction** on the degree of formulas as follows

(i) for any $a \in VAR$, $v^*(a) = v(a)$;

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B)$$

Ł Semantics: Satisfaction Relation

Step 3 of Ł-semantics **definition**

Definition

Let $v : VAR \longrightarrow \{T, \perp F\}$

We say that a truth assignment v **Ł - satisfies** a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models_L A$

Definition

We say that a truth assignment v **does not Ł- satisfy** a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models_L A$

Ł - Semantics: Model, Counter Model

Ł - Model

Any truth assignment v, $v : VAR \longrightarrow \{F, \perp, T\}$ such that

$$v \models_L A$$

is called a **Ł - model** for A

Ł - Counter Model

Any v such that

$$v \not\models_L A$$

is called a **Ł - counter model** for the formula A

Ł - Semantics: Tautology

Step 4 of Ł-semantics **definition**

Definition

For any $A \in \mathcal{F}$,

A is a **Ł tautology** if and only if

$v^*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$

We also say that

A is a **Ł tautology** if and only if all truth assignments

$v : VAR \longrightarrow \{F, \perp, T\}$ are **Ł models** for A

Notation

$$\models_L A$$

Set of all Ł tautologies

$$\mathbf{LT} = \{A \in \mathcal{F} : \models_L A\}$$

Ł Tautologies

Let

LT, **T** denote the sets of all Ł tautologies and the classical tautologies, respectively.

Q1 Is the Ł logic (defined semantically!) really **different** from the **classical logic**?

It means are theirs sets of tautologies different?

Answer: **YES**, they are different sets.

Consider a classical tautology $(\neg a \cup a)$, i.e. we know that

$$\models (\neg a \cup a)$$

We will show that

$$\not\models_L \ (\neg a \cup a)$$

Classical and Ł Tautologies

Consider the formula $(\neg a \cup a)$

Take a truth assignment v such that

$$v(a) = \perp$$

Evaluate

$v^*(\neg a \cup a) = v^*(\neg a) \cup v^*(a) = \neg v(a) \cup v(a)$

$= \neg \perp \cup \perp = \perp \cup \perp = \perp$

This proves that v is a counter-model for $(\neg a \cup a)$

$$\not\models_L \; (\neg a \cup a)$$

and we have a property:

$$\mathbf{LT} \neq \mathbf{T}$$

Classical and Ł Tautologies

Q2 Do the Ł logic and classical logic have something more in common besides the common language?

Do they share some tautologies?

Which is the relationship (if any) between their sets of tautologies **LT** and **T**?

Answer

YES, they do share tautologies and

YES, they do have an interesting relationship

Classical and Ł Tautologies

Let's **restrict** the functions defining Ł connectives (Truth Tables for Ł connectives) to the values T and F

Observe that by doing so we get the Truth Tables for classical connectives, i.e. the following holds for any $A \in \mathcal{F}$

If $v^*(A) = T$ for all $v : VAR \longrightarrow \{F, \perp, T\}$,

then $v^*(A) = T$ for all $v : VAR \longrightarrow \{F, T\}$

We have hence **proved** that

$$\mathbf{LT} \subset \mathbf{T}$$

Łukasiewicz Life, Works and Logics

Jan Leopold Łukasiewicz was born on 21 December 1878 in Lwow, historically a Polish city, at that time the capital of Austrian Galicia

He died on 13 February 1956 in Ireland and is buried in Glasnevin Cemetery in Dublin, " far from dear Lwow and Poland ", as his gravestone reads

Here is a very good, interesting and extended entry in Stanford Encyclopedia of Philosophy about his life, influences, achievements, and logics

http://plato.stanford.edu/entries/lukasiewicz/index.html

Kleene Logic K

Motivation

We model now a situation where the third logical value $\perp$ intuitively represents the notion of "undecided", or "state of partial ignorance"

A sentence is assigned a value $\perp$ just in case it is not known to be either true or false

For example imagine a detective trying to solve a murder.

He may conjecture that Jones killed the victim. He cannot, at present, assign a truth value T or F to his conjecture, so we assign the value $\perp$

But it is certainly either true of false and $\perp$ represents our ignorance rather then total unknown.

K - Language

The **K** - Language is the same in case of classical propositional and L logic, i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

We form the set $\mathcal{F}$ of formulas in a standard way

K- Semantics: Connectives

Connectives $\neg, \cup, \cap$ of **K** are defined as in **Ł** semantics, i.e.

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F$$

and for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \cup y = max\{x, \ y\}$$

$$x \cap y = min\{x, \ y\}$$

Remember that we assumed: $F < \perp < T$

K- Semantics: Connectives

Implication

For any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \Rightarrow y = \neg x \cup b$$

Kleene's 3-valued truth tables **differ** hence from Łukasiewicz's truth tables only in a case of **implication.** This table is:

K-Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	$\perp$	T
T	F	$\perp$	T

K Tautologies

$$\mathbf{KT} = \{A \in \mathcal{F} : \models_K A\}$$

Relationship between Ł, K, and classical logic.

$$\mathbf{LT} \neq \mathbf{KT},$$

$$\mathbf{KT} \subset \mathbf{T}$$

Proof of $\mathbf{LT} \neq \mathbf{KT}$.
Obviously $\models_L (a \Rightarrow a)$
Take v such that $v(a) = \perp$; we have that for **K** semantics
$v^*(a \Rightarrow a) = (v(a) \Rightarrow v(a)) = (\perp \Rightarrow \perp) = \perp$
This **proves** that $\not\models_K (a \Rightarrow a)$
and

$$\mathbf{LT} \neq \mathbf{KT}$$

K- Tautologies

The second sets of tautologies property

$$\mathbf{KT} \subset \mathbf{T}$$

follows directly from the the fact that, as in the **Ł** case, if we restrict the **K** Truth Tables to the values *T* and *F* only, we get the Truth Tables for classical connectives

Heying Logic H

Motivation and History

We call the **H** logic also a Heyting logic because its connectives are defined as operations on the set $\{F, \perp, T\}$ in such a way that they form a 3-element **pseudo-Boolean algebra** which is also often called a 3-element **Heyting algebra**

Pseudo-Boolean algebras were invented and developed as the first ever semantics for the Intuitionistic Logic

Motivation and History

The Intuitionistic Logic was defined by its inventor Brouwer and his school in 1900s as a **proof system** only.

Heyting provided **first axiomatization** for the Intuitionistic Logic, so the **pseudo-Boolean algebras** are often also called Heyting algebras in his honor

The **pseudo-Boolean** algebras semantics was discovered some 35 years later by McKinsey and Tarski in 1942 for **Intuitionistic propositional logic** only

It took yet another 15 years to extend it to predicate Intuitionistic logic by Rasiowa, Mostowski in 1957

Motivation and History

A formula *A* is an **Intuitionistic tautology** if and only if it is true in all **pseudo-Boolean algebras**

Hence, if *A* is an **Intuitionistic tautology** , it is also a tautology under the 3- valued Heyting semantics

If A **is not** a 3- valued Heyting **tautology**, then **it is not** an Intuitionistic **tautology**

It means that our 3-valued **Heyting semantics** is a good candidate for a **counter model** for the formulas that **might not be** Intuitionistic **tautologies**

H Logic and Intuitionistic Logic

Denote by **IT**, **HT** the sets of all **tautologies** of the Intuitionistic logic and Heyting 3-valued logic, respectively .

We have that

$$\mathbf{IT} \subset \mathbf{HT}$$

We conclude that for any formula A,

$$\text{If} \quad \not\models_H \ A \quad \text{then} \quad \not\models_I \ A$$

It means that if we show that a formula A has a Heying 3-valued counter-]model, then we have proved that it is not an intuitionistic tautology.

Kripke Models

The other type of semantics for the Intuitionistic Logic were defined by **Kripke** in 1964

They are called **Kripke Models**

Kripke Models were proved to be equivalent to the **pseudo-Boolean algebras** models in case of the Intuitionistic Logic

Kripke Models are very general and serve as a general method of defining not extensional semantics for various classes of logics

That includes **semantics** for hundreds of Modal, Knowledge Logic and different logics developed and being developed by computer scientists

H Semantics

The Language:

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

Logical connectives: $\cup$ and $\cap$ are the same as in the case of **Ł** and **K** semantics , i.e.

for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \cup y = max\{x,\ y\}, \qquad x \cap = min\{x,\ y\}$$

Remember that we assumed: $F < \perp < T$

Heyting Semantics

Implication

$$\Rightarrow: \quad \{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$$

such that for any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation

$$\neg x = x \Rightarrow F$$

H Truth Tables

H Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

H Negation

$\neg$	F	$\perp$	T
	T	*F*	F

Sets of Tautologies Relationships

Notation: **HT**, **T**, **LT**, **KT** denote the set of all tautologies of the H, classical, Ł, and K logic, respectively.

Relationships:

$$\mathbf{HT} \neq \mathbf{T} \neq \mathbf{LT} \neq \mathbf{KT},$$

$$\mathbf{HT} \subset \mathbf{T}$$

Proof of $\mathbf{HT} \neq \mathbf{T}$

For the formula $(\neg a \cup a)$ we have:

$$\models (\neg a \cup a) \quad \text{and} \quad \not\models_H (\neg a \cup a)$$

Sets of Tautologies Relationships

Proof of **HT** $\neq$ **KT**

Take any truth assignment v , such that $v^*(A) = \perp$

We get

$$\models_H (A \Rightarrow A)$$

but

$$\not\models_K (A \Rightarrow A)$$

Sets of Tautologies Relationships

Proof of **HT** $\neq$ **LT**

Take now a variable assignment v such that $v(a) = v(b) = \perp$

It proves that

$$\not\models_K (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

but we verify that

$$\models_L (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

Sets of Tautologies Relationships

Proof of **$\mathbf{HT} \subset \mathbf{T}$**

Observe now that if we restrict the truth tables for H to connectives T and F only,

we get the truth tables for classical connectives.

All together we have **proved** that the classical logic **extends** all of our three-valued logics L, K and H, i.e.

$$\mathbf{LT} \subset \mathbf{T}, \quad \mathbf{KT} \subset \mathbf{T}, \quad \mathbf{HT} \subset \mathbf{T}$$

Bochvar 3-valued logic B

Motivation

Consider a semantic paradox given by a sentence: this sentence is false.

If it is true it must be false,

if it is false it must be true.

According to Bochvar, such sentences are neither true of false but rather paradoxical or meaningless.

Bochvar 3-valued logic B

Bochvar's semantics follows the principle that the third logical value, denoted now by m (for mining less) is in some sense "infectious";

if one component of the formula is assigned the value m then the formula is also assigned the value m.

Bochvar also adds an one assertion operator S that asserts the logical value of T and F , i.e.

$$SF = F, \quad SF = F$$

and it asserts that meaningfulness m is false, i.e

$$Sm = F$$

B Language

Language: we add a new one argument connective S and get

$$\mathcal{L}_B = \mathcal{L}_{\{\neg, S, \Rightarrow, \cup, \cap\}}$$

We denote by $\mathcal{F}_B$ the set of all formulas of the language $\mathcal{L}_B$ and by $\mathcal{F}$ the set of formulas of the language $\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$ common to the classical and all 3 valued logics considered till now.

Observe that directly from the definition we have that

$$\mathcal{F} \subset \mathcal{F}_{\mathcal{B}}$$

The formula SA reads "assert A"

B Logical Connectives

B Negation

$\neg$	F	m	T
	F	m	T

B Conjunction

$\cap$	F	m	T
F	F	m	F
m	m	m	m
T	F	m	T

B Semantics

B Disjunction

$\cup$	F	m	T
F	F	m	T
m	m	m	m
T	T	m	T

B Implication

$\Rightarrow$	F	m	T
F	T	m	T
m	m	m	m
T	F	m	T

B Assertion

S	F	m	T
	F	F	T

B Tautologies

$$\mathbf{BT} = \{A \in \mathcal{F}_B : \ \models_B A\}$$

Let A be a formula that do not contain the assertion operator S, i.e. the formula $A \in \mathcal{F}$ of the language $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$

Observe that any v, such that v(a) = m for at least one variable in the formula $A \in \mathcal{F}$ is a counter-model for that formula.

So we have that $\mathbf{T} \cap \mathbf{BT} = \emptyset$

Observe that for a formula $A \in \mathcal{F}_B$ to be a **B tautology**, it must contain the connective S in front of each variable appearing in A

Chapter 5
Some Simple Review Problems

Exercise 1

Reminder: we define H semantics operations $\cup$ and $\cap$ as follows

For any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put

$$x \cup y = max\{x,\ y\}, \quad x \cap y = min\{x,\ y\}$$

Implication :

$$x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Negation:

$$\neg a = a \Rightarrow F.$$

Exercise 1

Question We know that

$$v : VAR \longrightarrow \{F, \perp, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$$

under H semantics.

Evaluate

$$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$$

Exercise 1

Solution

$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under H semantics

if and only if $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$

if and only if $a = T, b = T$ and $(T \Rightarrow c) = \perp$

if and only if $c = \perp$.

I.e. we have that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$

if and only if $a = T,\ b = T,\ c = \perp$

Exercise 1

Now we can we **evaluate**

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$

$= (((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T))$

$= ((T \Rightarrow (T \Rightarrow F)) \cup T)$

$=$ T

Exercise 2

We **define** a 4 valued $\mathbf{L}_4$ logic semantics as follows.

The language is $\mathcal{L}_{\{\neg,\Rightarrow,\cup,\cap\}}$

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ as the following operations in the set

$\{F, \perp_1, \perp_2, T\}$, where $F < \perp_1 < \perp_2 < T$

Negation

$\neg : \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$,

such that

$$\neg\perp_1 = \perp_1, \ \neg\perp_2 = \perp_2, \ \neg F = T, \ \neg T = F$$

Exercise 2

Conjunction

$\cap : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \ \perp_1, \perp_2, , T\}$

such that for any $(x, y) \in \{T, \perp_1, \perp_2, F\} \times \{T, \perp_1, \perp_2, F\}$ we put

$$x \cap y = min\{x, y\}$$

Disjunction

$\cup : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$

such that for any $(x, y) \in \{T, \perp_1, \perp_2, F\} \times \{T, \perp_1, \perp_2, F\}$ we put

$$x \cup y = max\{x, y\}$$

Exercise 2

Implication

$\Rightarrow : \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$

such that for any $(x, y) \in \{T, \perp_1, \perp_2, F\} \times \{T, \perp_1, \perp_2, F\}$ we put

$$x \Rightarrow y = \begin{cases} \neg x \cup y & \text{if } x > y \\ T & \text{otherwise} \end{cases}$$

Verify whether

$$\models_4 ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Exercise 2

Solution

Let v be a truth assignment such that $v(a) = v(b) = \perp_1$

We **evaluate**

$v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg\perp_1 \cup \perp_1))$

$= (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1.$

This **proves** that v is a counter-model for our formula and

$$\not\models_4 ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Exercise 2

Observe that a v such that

$v(a) = v(b) = \perp_2$ is also a counter model

We evaluate (in shorthand notation)

$v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg \perp_2 \cup \perp_2))$

$= (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2)$

$= \perp_2$

CHAPTER 5

PART 2: Many Valued Extensional Semantics **M**

Many Valued Extensional Semantics **M**

Here is a straightforward **generalization** of classical and 3 valued semantics presented here to a semantics **M** defined for any propositional language

The semantics **M** defined here is **extensional** and is defined for a non-empty set of V of logical values of any cardinality

We only assume that the set V of logical values of **M** always has a special, distinguished logical value which serves to define a notion of tautology

We denote this distinguished value as T

Many Valued Extensional Semantics **M**

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Formal definition of **many valued extensional semantics M** for the language $\mathcal{L}_{CON}$ follows the pattern of the classical and 3-valued cases and consists of giving **definitions** of the following main components:

1. Extensional l Connectives under semantics **M**
2. Truth Assignment for **M**
3. Satisfaction Relation, Model, Counter-Model under semantics **M**
4. Tautology under semantics **M**

Definition of **M** - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Let V be a non-empty set of **logical values** adopted by the semantics **M**

We adopt now a following formal definition of **M** - extensional connectives

Definition

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called **M -extensional** iff their semantics **M** is defined by respective functions

$$\nabla : \; V \longrightarrow V \quad \text{and} \quad \circ : \; V \times V \longrightarrow V$$

Definability of Connectives under a semantics M

Given a propositional language $\mathcal{L}_{CON}$ and its **extensional semantics M**

We adopt the following definition

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ...\circ_n \in CON$ for $n \geq 1$ **under the semantics M** if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ...\circ_n$ as they are **defined by the semantics M**

Example

Classical implication $\Rightarrow$ is **definable** in terms of $\cup$ and $\neg$ **under classical semantics** because under this semantics $\Rightarrow$ is a **composition** of functions $\neg$ and $\cup$ defined as follows

For all $(a, b) \in \{T, F\} \times \{T, F\}$,

$$a \Rightarrow b = \neg a \cup b$$

Definability of Connectives

Exercise 1

Verify which (if any) of our 3 valued semantics **L, K, H** $\cap, \cup$ are **definable** in terms of $\Rightarrow$ and $\neg$ by the classical case composition formula $a \Rightarrow b = \neg a \cup b$

Exercise 2

Verify which of our 3 valued semantics **L, K, H** $\cup$ is **definable** in terms of $\Rightarrow$ alone

M Semantics Assumption

We assume that the set V of logical values of **M** always has a special, distinguished logical value which serves to define a notion of tautology under the semantics **M**

We denote this distinguished value as T

Step 2

We define **M semantics**, as in previous cases, in terms of the propositional connectives as defined in the **Step 1** and a function called **M** truth assignment

Definition

M truth assignment is **any function**

$$v : VAR \longrightarrow V$$

where V is the set of logical values of **M**

M Truth Assignment Extension v^* to $\mathcal{F}$

Definition

Given the **M** truth assignment

$$v : VAR \longrightarrow V$$

We define its **M extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function

$$v^* : \mathcal{F} \longrightarrow V$$

such that the following conditions are satisfied

(i) for any $a \in VAR$

$$v^*(a) = v(a);$$

M Truth Assignment Extension v^* to $\mathcal{F}$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

The symbols on the **left-hand side** of the equations represent connectives in their natural language meaning and

the symbols on the **right-hand side** represent connectives in their semantical meaning as defined by the **semantics M**

M Semantics: Satisfaction Relation

Step 3

Definition: Let $v : VAR \longrightarrow V$

Let $T \in V$ be the **distinguished logical value**

We say that

v **M** satisfies a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models_{\mathbf{M}} A$

Definition: We say that

v does not **M** satisfy a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models_{\mathbf{M}} A$

The relation $\models_{\mathbf{M}}$ is called a **satisfaction relation** under semantics **M**, or **M satisfaction relation** for short

M Semantics: Model, Counter-Model

Definition:

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow V$

Any v such that $v \models_{\mathbf{M}} A$ is called a **M model** for A

Any v such that $v \not\models_{\mathbf{M}} A$ is called a **M counter model** for A

M Semantics: Tautology

Step 4

Definition:

For any formula $A \in \mathcal{F}$

A is a **M tautology** iff $v^*(A) = T$, **for all** $v : VAR \longrightarrow V$

i.e. we have that

A is a **M tautology** iff **any** $v : VAR \longrightarrow V$ is a **M model** for A

Notation

We write symbolically $\models_{\mathbf{M}} A$ for the statement "A is a **M** tautology"

Semantics: not a tautology

Definition

A is **not a M tautology** iff there is v, such that $v^*(A) \neq T$

i.e. we have that

A is **not a M tautology** iff A has a **M counter-model**

Notation

We write $\not\models_{\mathbf{M}} A$ to denote the statement "A is not **M** tautology"

Challenge Exercise

1. Define your own propositional language $\mathcal{L}_{CON}$ that contains also **different connectives** that the standard connectives $\neg, \cup, \cap, \Rightarrow$

Your language $\mathcal{L}_{CON}$ **does not need** to include all (if any!) of the standard connectives $\neg, \cup, \cap, \Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for **your own semantic**

4. Define formally your own **extensional semantics M** for your language $\mathcal{L}_{CON}$ - it means

write carefully all **Steps 1- 4** of the definition of your **M**