cse371/mat371
LOGIC

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LECTURE 3a

Chapter 3
Classical Propositional Semantics

Semantics- General Principles

Given a propositional language $\mathcal{L} = \mathcal{L}_{CON}$

Symbols for connectives of $\mathcal{L}$ always have some intuitive meaning

Semantics provides a formal definition of the meaning of these symbols

It also provides a method of defining a notion of a **tautology**, i.e. of a formula of the language that is always true under the given semantics

Extensional Connectives

In **Chapter 2** we described the intuitive classical propositional semantics and its motivation and introduced the following notion of extensional connectives

Extensional connectives are the propositional connectives that have the following property:
the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas

We also assumed that

All classical **propositional connectives**

$$\neg,\ \cup,\ \cap,\ \Rightarrow,\ \Leftrightarrow,\ \uparrow,\ \downarrow$$

are **extensional**

Non-Extensional Connectives

We have also observed the following

Remark

In everyday language there are expressions such as

"I believe that", "it is possible that", " certainly", etc....

They are represented by some **propositional connectives** which **are not extensional**

Non- extensional connectives do not play any role in **mathematics** and so are not discussed in **classical logic** and will be studied separately

Definition of Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Let V be a non-empty set of **logical values**

We adopt now a following formal definition of extensional connectives

Definition

Connectives $\nabla \in C_1$, $\circ \in C_2$ are called **extensional** iff their semantics is defined by respective functions

$$\nabla : V \longrightarrow V \quad \text{and} \quad \circ : V \times V \longrightarrow V$$

Functional Dependency and Definability of Connectives

In **Chapter 2** we talked about **functional dependency** of connectives and of **definability** of a connective in terms of other connectives

We define these notions formally as follows

Given a propositional language $\mathcal{L}_{CON}$ and an **extensional semantics** for it; i.e a semantics such that all connectives in $\mathcal{L}$ are **extensional**

Definition

Connectives $\circ \in CON$ and $\circ_1, \circ_2, ...\circ_n \in CON$ (for $n \geq 1$) are **functionally dependent** iff $\circ$ is a **certain function composition** of functions $\circ_1, \circ_2, ...\circ_n$

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ...\circ_n \in CON$ iff $\circ \in CON$ and $\circ_1, \circ_2, ...\circ_n \in CON$ are **functionally dependent**

Classical Propositional Semantics Assumptions

Assumptions

A1: We define our semantics for the language

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow,\ \Leftrightarrow\}}$$

A2: **Two values:** the set of logical values $V = \{T,\ F\}$

Logical values T, F **denote** truth and falsehood, respectively

There are other notations, for example 0,1

A3: **Extensionality:** all connectives of $\mathcal{L}$ are **extensional**

Semantics for any language $\mathcal{L}$ for which the assumption **A3** holds is called **extensional semantics**

Propositional Semantics Definition

Formal definition of a propositional **extensional semantics** for a given language $\mathcal{L}_{CON}$ consists of providing **definitions** of the following four main components:

1. Extensional Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology

The definition of the **classical semantics** and **extensional semantics** for some non-classical logics considered here will follow **the same pattern**

Semantics: Classical Connectives Definition

Semantics Definition **Step 1**

The assumption of **extensionality of connectives** means that unary connectives are **functions** defined on a set $\{T, F\}$ with values in the set $\{T, F\}$ and

binary connectives are **functions** defined on a set $\{T, F\} \times \{T, F\}$ with values in the set $\{T, F\}$

In particular we adopt the following definitions

Negation Definition

Negation $\neg$ is a **function:**

$$\neg : \{T, F\} \longrightarrow \{T, F\},$$

such that

$$\neg T = F, \quad \neg F = T$$

Notation

When defining connectives as functions we usually write the name of a function (our connective) **between the arguments**, not in front as in function notation, i.e. for example we write $T \cap T = T$ instead of $\cap(T, T) = T$

Conjunction Definition

Conjunction $\cap$ is a **function:**

$$\cap : \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that

$$\cap(T, T) = T, \quad \cap(T, F) = F, \quad \cap(F, T) = F, \quad \cap(F, F) = F$$

We write it as

$$T \cap T = T, \quad T \cap F = F, \quad F \cap T = F, \quad F \cap F = F$$

Semantics: Classical Connectives Definition

Disjunction Definition

Disjunction $\cup$ is a **function:**

$$\cup : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$$

such that

$$\cup(T, T) = T, \quad \cup(T, F) = T, \quad \cup(F, T) = T, \quad \cup(F, F) = F$$

We write it as

$$T \cup T = T, \quad T \cup F = T, \quad F \cup T = T, \quad F \cup F = F$$

Semantics: Classical Connectives Definition

Implication Definition

Implication $\Rightarrow$ is a **function:**

$$\Rightarrow : \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$$

such that

$$\Rightarrow (T, T) = T, \quad \Rightarrow (T, F) = F, \quad \Rightarrow (F, T) = T, \quad \Rightarrow (F, F) = T$$

We write it as

$$T \Rightarrow T = T, \quad T \Rightarrow F = F, \quad F \Rightarrow T = T, \quad F \Rightarrow F = T$$

Equivalence Definition

Equivalence $\Leftrightarrow$ is a **function:**

$$\Leftrightarrow: \ \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$$

such that

$$\Leftrightarrow (T, T) = T, \quad \Leftrightarrow (T, F) = F, \quad \Leftrightarrow (F, T) = F, \quad \Leftrightarrow (T, T) = T$$

We write it as

$$T \Leftrightarrow T = T, \quad T \Leftrightarrow F = F, \quad F \Leftrightarrow T = F, \quad T \Leftrightarrow T = T$$

Classical Connectives Truth Tables

We write the functions defining connectives in a form of tables, usually called the classical truth tables

Negation:

$$\neg T = F, \quad \neg F = T$$

$\neg$	T	F
	F	T

Conjunction:

$$T \cap T = T, \quad T \cap F = F, \quad F \cap T = F, \quad F \cap F = F$$

$\cap$	T	F
T	T	F
F	F	F

Classical Connectives Truth Tables

Disjunction:

$$T \cup T = T, \quad T \cup F = T, \quad F \cup T = T, \quad F \cup F = F$$

$\cup$	T	F
T	T	T
F	T	F

Implication:

$$T \Rightarrow T = T, \quad T \Rightarrow F = F, \quad F \Rightarrow T = T, \quad F \Rightarrow F = T$$

$\Rightarrow$	T	F
T	T	F
F	T	T

Classical Connectives Truth Tables

Equivalence:

$$T \Leftrightarrow T = T,\; T \Leftrightarrow F = F,\; F \Leftrightarrow T = F,\; F \Leftrightarrow F = T$$

$\Leftrightarrow$	T	F
T	T	F
F	F	T

This ends the **Step1** of the semantics definition

Definability of Classical Connectives

We adopted the following definition

Definition

A connective $\circ \in CON$ is **definable** in terms of some connectives $\circ_1, \circ_2, ...\circ_n \in CON$ iff $\circ$ is a **certain function composition** of functions $\circ_1, \circ_2, ...\circ_n$

Example

Classical implication $\Rightarrow$ is **definable** in terms of $\cup$ and $\neg$ because $\Rightarrow$ can be defined as a **composition** of functions $\neg$ and $\cup$

More precisely, a function $h : \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ defined by a formula

$$h(x, y) = \cup(\neg x, y)$$

is a **composition of functions** $\neg$ and $\cup$ and we **prove** that the implication function $\Rightarrow$ is equal with h

Short Review: Equality of Functions

Definition

Given two sets A, B and functions f, g such that

$$f:\ A \longrightarrow B \quad \text{and} \quad g:\ A \longrightarrow B$$

We say that the functions f, g are **equal** and write is as f = g iff f(x) = g(x) for all elements $x \in A$

Example: Consider functions

$$\Rightarrow:\ \{T,F\} \times \{T,F\} \longrightarrow \{T,F\} \quad \text{and} \quad h : \{T,F\} \times \{T,F\} \longrightarrow \{T,F\}$$

where $\Rightarrow$ is classical implication and h is defined by the formula $h(x,y) = \cup(\neg x, y)$

We **prove** that $\Rightarrow\ =\ h$ by evaluating that
$\Rightarrow (x,y) = h(x,y) = \cup(\neg x, y)$, for all $(x,y) \in \{T,F\} \times \{T,F\}$

Definability of Classical Implication

We re-write formula $\Rightarrow (x, y) = \cup(\neg x, y)$ in our adopted notation as

$$x \Rightarrow y = \neg x \cup y \quad \text{for all} \quad (x, y) \in \{T, F\} \times \{T, F\}$$

and call it a **formula defining** $\Rightarrow$ in terms of $\cup$ and $\neg$

We verify correctness of the **definition** as follows

$T \Rightarrow T = T$ and $\neg T \cup T = F \cup T = T$ **yes**

$T \Rightarrow F = F$ and $\neg T \cup F = F \cup F = F$ **yes**

$F \Rightarrow F = T$ and $\neg F \cup F = T \cup F = T$ **yes**

$F \Rightarrow T = T$ and $\neg F \cup T = T \cup T = T$ **yes**

Definability of Classical Connectives

Exercise 1

Find a **formula defining** $\cap, \Leftrightarrow$ in terms of $\cup$ and $\neg$

Exercise 2

Find a **formula defining**
$\Rightarrow, \cup, \Leftrightarrow$ in terms of $\cap$ and $\neg$

Exercise 3

Find a **formula defining** $\cap, \cup, \Leftrightarrow$ in terms of $\Rightarrow$ and $\neg$

Exercise 4

Find a **formula defining** $\cup$ in terms of $\Rightarrow$ alone

Two More Classical Connectives

Sheffer Alternative Negation $\uparrow$

$$\uparrow: \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$$

such that

$$T \uparrow T = F, \quad T \uparrow F = T, \quad F \uparrow T = T, \quad F \uparrow F = T$$

Łukasiewicz Joint Negation $\downarrow$

$$\downarrow: \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$$

such that

$$T \downarrow T = F, \quad T \downarrow F = F, \quad F \downarrow T = F, \quad F \downarrow F = T$$

Definability of Classical Connectives

Exercise 4

Show that the **Sheffer Alternative Negation** $\uparrow$ defines all classical connectives $\neg, \Rightarrow, \cup, \cap, \Leftrightarrow$

Exercise 5

Show that **Łukasiewicz Joint Negation** $\downarrow$ defines all classical connectives $\neg, \Rightarrow, \cup, \cap, \Leftrightarrow$

Exercise 6

Show that the two binary connectives: $\downarrow$ and $\uparrow$ suffice, each of them separately, to define **all classical connectives**, whether unary or binary

Semantics: Truth Assignment

Step 2

We define the next components of the classical propositional **semantics** in terms of the propositional connectives as defined in the **Step 1** and a function called truth assignment

Definition

A **truth assignment** is **any function**

$$v : VAR \longrightarrow \{T, F\}$$

Observe that the domain of **truth assignment** is the set of propositional variables, i.e. the **truth assignment** is defined only for **atomic formulas**

Truth Assignment Extension

We now **extend** the truth assignment v to the set of **all formulas** $\mathcal{F}$ in order **define formally** the logical value for **any formula** $A \in \mathcal{F}$

The definition of the **extension** of the variable assignment v to the set $\mathcal{F}$ follows the same pattern for the all extensional connectives, i.e. for **all extensional semantics**

Truth Assignment Extension v^* to $\mathcal{F}$

Definition

Given the truth assignment

$$v : VAR \longrightarrow \{T, F\}$$

We define its **extension** v^* to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function

$$v^* : \mathcal{F} \longrightarrow \{T, F\}$$

such that the following conditions are satisfied

(i) for any $a \in VAR$

$$v^*(a) = v(a);$$

Truth Assignment Extension v^* to $\mathcal{F}$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = \cap(v^*(A), v^*(B));$$

$$v^*((A \cup B)) = \cup(v^*(A), v^*(B));$$

$$v^*((A \Rightarrow B)) = \Rightarrow(v^*(A), v^*(B));$$

$$v^*((A \Leftrightarrow B)) = \Leftrightarrow(v^*(A), v^*(B))$$

The symbols on the **left-hand side** of the equations represent connectives in their **natural language meaning** and

the symbols on the **right-hand side** represent connectives in their **semantical meaning** given by the classical truth tables

Extension v^* Definition Revisited

Notation

For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations

The **condition (ii)** of the definition of the extension v^* can be hence **written** as follows

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*((A \cap B)) = v^*(A) \cap v^*(B);$$

$$v^*((A \cup B)) = v^*(A) \cup v^*(B);$$

$$v^*((A \Rightarrow B)) = v^*(A) \Rightarrow v^*(B);$$

$$v^*((A \Leftrightarrow B)) = v^*(A) \Leftrightarrow v^*(B)$$

We will use this notation for the rest of the book

Truth Assignment Extension Example

Consider a formula

$$((a \Rightarrow b) \cup \neg a))$$

and a truth assignment v such that

$$v(a) = T, \quad v(b) = F$$

Observe that we did not specify $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$ of the formula A

We say: "v such that" - as we consider its values for the set $\{a, b\} \subseteq VAR$

Nevertheless, the domain of v is the set of all variables VAR and we have to **remember** that.

Truth Assignment Extension Example

Given a formula A: $((a \Rightarrow b) \cup \neg a))$ and a truth assignment v such that v(a) = T, v(b) = F

We calculate the logical value of the formula A as follows:

$v^*(A) = v^*(((a \Rightarrow b) \cup \neg a))) = \cup(v^*((a \Rightarrow b), v^*(\neg a))=$

$\cup(\Rightarrow(v^*(a), v^*(b)), \neg v^*(a))) = \cup(\Rightarrow(v(a), v(b)), \neg v(a)))=$

$\cup(\Rightarrow(T, F), \neg T)) = \cup(F, F) = F$

We can also calculate it as follows:

$v^*(A) = v^*(((a \Rightarrow b) \cup \neg a))) = v^*((a \Rightarrow b)) \cup v^*(\neg a) =$

$(v(a) \Rightarrow v(b)) \cup \neg v(a) = (T \Rightarrow F) \cup \neg T = F \cup F = F$

We write it in a **short-hand notation** as

$(T \Rightarrow F) \cup \neg T = F \cup F = F$

On **tests** I will specify when you can use the the **short-hand notation**.

Semantics: Satisfaction Relation

Step 3

Definition: Let $v : VAR \longrightarrow \{T, F\}$

We say that
v **satisfies** a formula $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$

Definition: We say that
v **does not satisfy** a formula $A \in \mathcal{F}$ iff $v^*(A) \neq T$

Notation: $v \not\models A$

The relation $\models$ is called a **satisfaction relation**

Semantics: Satisfaction Relation

Observe that $v^*(A) \neq T$ is is equivalent to the fact that $v^*(A) = F$ **only** in 2-valued semantics and

$$v \not\models A \quad \text{iff} \quad v^*(A) = F$$

Definition

We say that v **falsifies** the formula A iff $v^*(A) = F$

Remark

For any formula $A \in \mathcal{F}$

$v \not\models A$ iff v **falsifies** the formula A

Examples

Example 1 : Let $A = ((a \Rightarrow b) \cup \neg a))$ and $v : VAR \longrightarrow \{T, F\}$ be such that $v(a) = T, v(b) = F$

We calculate $v^*(A)$ using a **short hand notation** as follows

$$(T \Rightarrow F) \cup \neg T = F \cup F = F$$

By definitiom

$$v \not\models ((a \Rightarrow b) \cup \neg a))$$

Observe that we did not need to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$

Examples

Example 2 Let $A = ((a \cap \neg b) \cup \neg c)$ and $v : VAR \longrightarrow \{T, F\}$ be such that
$v(a) = T, v(b) = F, v(c) = T$
We calculate $v^*(A)$ using a **short hand notation** as follows

$$(T \cap \neg F) \cup \neg T = (T \cap T) \cup F = T \cup F = T$$

By definition

$$v \models ((a \cap \neg b) \cup \neg c)$$

Examples

Example 3 Let $A = ((a \cap \neg b) \cup \neg c)$

Consider now $v_1 : VAR \longrightarrow \{T, F\}$ such that

$v_1(a) = T, v_1(b) = F,\ v_1(c) = T$ and

$v_1(x) = F,$ for all $x \in VAR - \{a, b, c\}$

Observe that

$v(a) = v_1(a),\ \ v(b) = v_1(b),\ \ v(c) = v_1(c)$

Hence we get

$$v_1 \models ((a \cap \neg b) \cup \neg c)$$

Examples

Example 4 Let $A = ((a \cap \neg b) \cup \neg c)$

Consider now $v_2 : VAR \longrightarrow \{T, F\}$ such that

$v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T$ and

$v_1(x) = F,$ for all $x \in VAR - \{a, b, c, d\}$

Observe that

$v(a) = v_2(a), v(b) = v_2(b), v(c) = v_2(c)$

Hence we get

$$v_2 \models ((a \cap \neg b) \cup \neg c)$$

Semantics: Model, Counter-Model

Definition:

Given a formula $A \in \mathcal{F}$ and $v : VAR \longrightarrow \{T, F\}$

Any v such that $v \models A$ is called a **model** for A

Any v such that $v \not\models A$ is called a **counter model** for A

Observe that all truth assignments v, v_1, v_2 from our **Examples 2, 3, 4** are **models** for the same formula A

Semantics: Tautology

Step 4

Definition:

For any formula $A \in \mathcal{F}$

A is a **tautology** iff $v^*(A) = T$, **for all** $v : VAR \longrightarrow \{T, F\}$

i.e. we have that

A is a **tautology** iff **any** $v : VAR \longrightarrow \{T, F\}$ is a **model** for A

Notation

We write symbolically $\models A$ for the statement "A is a tautology"

Semantics: not a tautology

Definition

A is **not a tautology** iff there is v, such that $v^*(A) \neq T$

i.e. we have that

A is **not a tautology** iff A has a **counter-model**

Notation

We write $\not\models A$ to denote the statement "A is not a tautology"

How Many

We just saw from the **Examples 2, 3, 4** that given a model v for a formula A, we defined 2 other models for A

These models were identical with v on the variables in the formula A

Visibly we can keep constructing in a similar way more and more of such models

A natural question arises:

Given a **model** for a the formula A , **how many other models** for A can be constructed?

The same question can be asked about counter-models for A, if they exist

Challenge Problem

Challenge Problem : prove the following

Model Theorem

For any formula $A \in \mathcal{F}$,

If A has a **model** (counter- model), then it has uncountably many (exactly as many as real numbers) of **models** (counter-models)

How Many

Here is a more general question

Question

Given a formula $A \in \mathcal{F}$,

how many truth assignments we have to consider to prove that the formula A? is a **tautology**?

We prove that there are as many of such truth assignments as real numbers

But FORTUNATELY only a finite number of them is differs on the variables included in the formula A and we do have the following

Tautology DecidabilityTheorem

The notion of classical propositional tautology $\models A$ is **decidable**

Restricted Truth Assignments

To address and to answer these questions formally we first introduce some notations and definitions

Notation: for any formula A, we denote by VAR_A a set of all variables that appear in A

Definition: Given $v : VAR \longrightarrow \{T, F\}$, any function $v_A : VAR_A \longrightarrow \{T, F\}$ such that $v(a) = v_A(a)$ for all $a \in VAR_A$ is called a **restriction** of v to the formula A

Fact 1

For any formula A, any v, and its **restriction** v_A

$$v \models A \quad \text{iff} \quad v_A \models A$$

Restricted Model

Definition: Given a formula $A \in \mathcal{F}$, any function

$$w : \quad VAR_A \longrightarrow \{T, F\}$$

is called a truth assignment **restricted** to A

Definition Given a formula $A \in \mathcal{F}$

Any function

$$w : \quad VAR_A \longrightarrow \{T, F\} \quad \text{such that} \quad w^*(A) = T$$

is called a **restricted MODEL** for A

Example

Example

$$A = ((a \cap \neg b) \cup \neg c)$$

$$VAR_A = \{a, b, c\}$$

Truth assignment **restricted** to A is any function:

$$w : \quad \{a, b, c\} \longrightarrow \{T, F\}.$$

We use the following theorem to count all possible truth assignment **restricted** to A

Counting Functions

Counting Functions Theorem

For any finite sets A and B,

if the set A has n elements and B has m elements, then

there are m^n **possible functions** that map A into B

Proof by Mathematical Induction over m

Example:

There are $2^3 = 8$ truth assignments w **restricted** to

$$A = ((a \Rightarrow \neg b) \cup \neg c)$$

Counting Theorem

Counting Theorem

For any $A \in \mathcal{F}$, there are

$$2^{|VAR_A|}$$

possible truth assignments **restricted** to A

Example

Let $A = ((a \cap \neg b) \cup \neg c)$

All w **restricted** to A are listed in the table below

w	a	b	c	$w^*(A)$ computation	$w^*(A)$
w_1	T	T	T	$(T \Rightarrow T) \cup \neg T = T \cup F = T$	T
w_2	T	T	F	$(T \Rightarrow T) \cup \neg F = T \cup T = T$	T
w_3	T	F	F	$(T \Rightarrow F) \cup \neg F = F \cup T = T$	T
w_4	F	F	T	$(F \Rightarrow F) \cup \neg T = T \cup F = T$	T
w_5	F	T	T	$(F \Rightarrow T) \cup \neg T = T \cup F = T$	T
w_6	F	T	F	$(F \Rightarrow T) \cup \neg F = T \cup T = T$	T
w_7	T	F	T	$(T \Rightarrow F) \cup \neg T = F \cup F = F$	F
w_8	F	F	F	$(F \Rightarrow F) \cup \neg F = T \cup T = T$	T

$w_1, w_2, w_3, w_4 w_5, w_6, w_8$ are restricted models for A

w_7 is a restricted counter- model for A

Restrictions and Extensions

Given a formula A and $w : VAR_A \longrightarrow \{T, F\}$

Definition

Any function v, such that $v : VAR \longrightarrow \{T, F\}$ and $v(a) = w(a)$, for all $a \in VAR_A$ is called an **extension** of w to the set VAR of all propositional variables

Fact 2

For any formula A, any w **restricted** to A, and any of its **extensions** v

$$w \models A \quad \text{iff} \quad v \models A$$

Tautology and Decidability

By the definition of a tautology and **Facts 1, 2** we get the following

TautologyTheorem

$$\models \; A \quad \text{iff} \quad w \models A \quad \text{for all} \quad w : VAR_A \longrightarrow \{T, F\}$$

From above and the **Counting Theorem** we get

Tautology DecidabilityTheorem

The notion of **classical propositional tautology** $\models \; A$ is **decidable**

Tautology Verification

We just PROVED correctness of the well known

Truth Table Tautology Verification Method :

to verify whether $\models A$ list and evaluate all possible truth assignments w restricted to A and we have that

$\models A$ if all w evaluate to T

$\not\models A$ if there is one w that evaluates to F

Truth Table Example

Consider a formula A:

$$(a \Rightarrow (a \cup b))$$

We write the Truth Table:

w	a	b	$w^*(A)$ computation	$w^*(A)$
w_1	T	T	$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$	T
w_2	T	F	$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$	T
w_3	F	T	$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$	T
w_4	F	F	$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$	T

We evaluated that for all w **restricted** to A, i.e. all functions $w : VAR_A \longrightarrow \{T, F\}, \quad w \models A$

This proves by **TautologyTheorem**

$$\models (a \Rightarrow (a \cup b))$$

Tautology Verification

Imagine now that A has for example 200 variables.

To find whether A is a tautology by using the **Truth Table Method** one would have to evaluate 200 variables long expressions - not to mention that one would have to list 2^{200} **restricted** truth assignments

I want you to use now and later in case of many valued semantics a more intelligent (and much faster!) method called **Proof by Contradiction Method**

In fact, I will not accept the Truth Tables verifications on any TEST and students using it will get **0 pts** for the problem

Tautology - Proof by Contradiction Method

Proof by Contradiction Method:

In this method, in order to **prove** that $\models A$ we proceed as follows

We **assume** that $\not\models A$

We **work** with this assumption

If we get a **contradiction**, we have **proved** that $\not\models A$ is **impossible**

We hence **proved** $\models A$

If we do not get a **contradiction**, it means that the assumption $\not\models A$ is **true**, i.e.

we have **proved** that $\not\models A$

Tautology - Proof by Contradiction Method

Proof by Contradiction Method:

in order to verify whether $\models A$ one works backwards, trying to find a truth assignment v which makes a formula A false.

If we **find one**, it means that A is not a tautology

if we prove that it is **impossible**, i.e. we got a **contradiction**

it means that the formula is a tautology

Example

Let $A = (a \Rightarrow (a \cup b)$

Step 1: Assume that $\not\models A$, i.e. we write in a shorthand notion $A = F$

Step 2: We use shorthand notation to analyze **Strep 1**

$(a \Rightarrow (a \cup b)) = F$ **iff** $a = T$ and $(a \cup b) = F$

Step 3: Analyze **Step 2**

$a = T$ and $(a \cup b) = F$, i.e. $(T \cup b) = F$

This is **impossible** by the definition of $\cup$

We got a **contradiction**, hence

$$\models \ (a \Rightarrow (a \cup b))$$

Example

Observe that exactly the same reasoning proves that for any formulas $A, B \in \mathcal{F}$,

$\models (A \Rightarrow (A \cup B))$

The following formulas are also tautologies

$((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d))$

$(((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \Rightarrow (((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \cup ((a \Rightarrow \neg e)))$

because they are particular cases of $(A \Rightarrow (A \cup B))$

Set of all Tautologies

$$\mathbf{T} = \{A \in \mathcal{F} : \ \models A\}$$

Definition

A formula $A \in \mathcal{F}$ is called a **contradiction** if it does not have a model

Contradiction Notation: $=\mid A$

Directly from the definition we have that

$$=\mid A \quad \text{iff} \quad v \not\models A \ \text{ for all } \ v : VAR \longrightarrow \{T, F\}$$

Set of all Contradictions

$$\mathbf{C} = \{A \in \mathcal{F} : \ =\mid A\}$$

Examples

Tautology $(A \Rightarrow (B \Rightarrow A))$

Contradiction $(A \cap \neg A)$

Neither $(a \cup \neg b)$

Consider the formula $(a \cup \neg b)$

Any v such that $v(a) = T$ is a model for $(a \cup \neg b)$, so it is not a contradiction

Any v such that $v(a) = F, \; v(b) = T$ is a counter-model for $(a \cup \neg b)$ so $\not\models (a \cup \neg b)$

Simple Properties

Theorem 1 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) $A \in \mathbf{T}$

(2) $\neg A \in \mathbf{C}$

(3) For all v, $v \models A$

Theorem 2 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) $A \in \mathbf{C}$

(2) $\neg A \in \mathbf{T}$

(6) For all v, $v \not\models A$

Constructing New Tautologies

We now formulate and prove a theorem which describes validity of a method of constructing **new tautologies** from **given tautologies**

First we introduce some convenient notations.

Notation 1: for any $A \in \mathcal{F}$ we write

$$A(a_1, a_2, ...a_n)$$

to denote that $a_1, a_2, ...a_n$ are fall propositional variables appearing in A

Notation 2: let $A_1, ...A_n$ be any formulas, we write

$$A(a_1/A_1, ..., a_n/A_n)$$

to denote the result of **simultaneous replacement** (substitution) all variables $a_1, a_2, ...a_n$ in A by formulas $A_1, ...A_n$, respectively.

Constructing NewTautologies

Theorem For any formulas $A,\ A_1, ...A_n \in \mathcal{F}$,

IF $\models\ A(a_1, a_2, ...a_n)$ and $B = A(a_1/A_1, ..., a_n/A_n)$,

THEN $\models B$

Proof: Let $B = A(a_1/A_1, ..., a_n/A_n)$ and let $b_1, b_2, ...b_m$ be all propositional variables which occur in $A_1, ...A_n$

Given a truth assignment $v : VAR \longrightarrow \{T, F\}$, the values $v(b_1), v(b_2), ...v(b_m)$ define $v^*(A_1), ...v^*(A_n)$ and, in turn define $v^*(A(a_1/A_1, ..., a_n/A_n))$

Constructing NewTautologies

Let now $w : VAR \longrightarrow \{T, F\}$ be a truth assignment such that $w(a_1) = v^*(A_1),\ w(a_2) = v^*(A_2), ...w(a_n) = v^*(A_n)$.

Obviously, $v^*(B) = w^*(A)$.

Since $\models A$, $\ w^*(A) = T$, for all possible w,

hence $\ v^*(B) = w^*(A) = T$ for all truth assignments w and

we have $\quad \models B$

Models for Sets of Formulas

Consider $\mathcal{L} = \mathcal{L}_{CON}$ and let $\mathcal{S} \neq \emptyset$ be any non empty set of formulas of $\mathcal{L}$, i.e.

$$\mathcal{S} \subseteq \mathcal{F}$$

We adopt the following definition.

Definition

A truth truth assignment $v : VAR \longrightarrow \{T, F\}$

is a **model for the set** $\mathcal{S}$ of formulas if and only if

$$v \models A \quad \text{for } \textbf{all formulas } A \in \mathcal{S}$$

We write

$$v \models \mathcal{S}$$

to denote that v **is a model for the set** $\mathcal{S}$ of formulas

Counter- Models for Sets of Formulas

Similarly, we define a notion of a **counter-model**

Definition

A truth assignment $v : VAR \longrightarrow \{T, F\}$

is a **counter-model for the set** $\mathcal{S} \neq \emptyset$

of formulas if and only if

$$v \not\models A \quad \textbf{for some formula} \quad A \in \mathcal{S}$$

We write

$$v \not\models \mathcal{S}$$

to denote that v is a **counter- model** for the set $\mathcal{S}$ of formulas

Restricted Model for Sets of Formulas

Remark that the set $\mathcal{S}$ can be **infinite**, or **finite**

In a case when $\mathcal{S}$ is a **finite** subset of formulas we define, as before, a notion of restricted model and restricted counter-model.

Definition

Let $\mathcal{S}$ be a **finite** subset of formulas and $v \models \mathcal{S}$

Any restriction of the model v to the domain

$$VAR_{\mathcal{S}} = \bigcup_{A \in \mathcal{S}} VAR_A$$

is called a **restricted model** for $\mathcal{S}$

Restricted Counter - Model for Sets of Formulas

Definition

Any restriction of a **counter-model** v of a set $\mathcal{S} \neq \emptyset$ of formulas to the domain

$$VAR_{\mathcal{S}} = \bigcup_{A \in \mathcal{S}} VAR_A$$

is called a **restricted counter-model** for $\mathcal{S}$

Example

Example

Let $\mathcal{L} = \mathcal{L}_{\{\neg, \cap\}}$ and let

$$\mathcal{S} = \{a,\ (a \cap \neg b),\ c,\ \neg b\}$$

We have now $VAR_{\mathcal{S}} = \{a, b, c\}$

and $v : VAR_{\mathcal{S}} \longrightarrow \{T, F\}$ such that

$v(a) = T, v(c) = T, v(b) = F$ is a restricted model for $\mathcal{S}$

and $v : VAR_{\mathcal{S}} \longrightarrow \{T, F\}$ such that $v(a) = F$

is a restricted counter-model for $\mathcal{S}$

Models for Infinite Sets

The set $\mathcal{S}$ from the previous example was a finite set

Natural question arises:

Question

Give an example of an infinite set $\mathcal{S}$ that **has a model**

Give an example of an infinite set $\mathcal{S}$ that **does not have model**

Ex1 Consider set **T** of all **tautologies**

It is a countably infinite set and by definition of a tautology any v is a **model** for **T**, i.e. $v \models \mathbf{T}$

Ex2 Consider set **C** of all **contradictions**

It is a countably infinite set and

for any v, $v \not\models \mathbf{C}$ by definition of a contradiction, i.e. any any v is a **counter-model** for **C**

Challenge Problems

P1 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \neq \mathbf{T}$ and $\mathcal{S}$ **has a model**

P2 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \cap \mathbf{T} = \emptyset$ and $\mathcal{S}$ **has a model**

P3 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \neq \mathbf{C}$ and $\mathcal{S}$ **does not** have a **model**

P4 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \neq \mathbf{C}$ and $\mathcal{S}$ **has a counter model**

P5 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \cap \mathbf{C} = \emptyset$ and $\mathcal{S}$ **has a counter model**

Chapter 4: Consistent Sets of Formulas

Definition

A set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**, i.e. we have that

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if **there is** v such that $v \models \mathcal{G}$

Otherwise $\mathcal{G}$ is called **inconsistent**

HALF Challenge Problems

P6 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \neq \mathbf{T}$ and $\mathcal{S}$ is **consistent**

P7 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \cap \mathbf{T} = \emptyset$ and $\mathcal{S}$ is **consistent**

P8 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \neq \mathbf{C}$ and $\mathcal{S}$ is **inconsistent**

P9 Give an example of an infinite set $\mathcal{S}$, such that $\mathcal{S} \cap \mathbf{C} = \emptyset$ and $\mathcal{S}$ is **inconsistent**

Chapter 4: Independent Statements

Definition

A formula A is called **independent** from a set $\mathcal{G} \subseteq \mathcal{F}$ if and only if **there are** truth assignments v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \quad \text{and} \quad v_2 \models \mathcal{G} \cup \{\neg A\}$$

i.e. we say that a formula A is **independent** if and only if

$$\mathcal{G} \cup \{A\} \quad \text{and} \quad \mathcal{G} \cup \{\neg A\} \quad \text{are } \textbf{consistent}$$

Example

Example

Given a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Show that $\mathcal{G}$ is **consistent**

Solution

We have to find $v : VAR \longrightarrow \{T, F\}$ such that

$$v \models \mathcal{G}$$

It means that we need to **find** v such that

$$v^*((a \cap b) \Rightarrow b) = T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T$$

Consistent: Example

1. Formula $((a \cap b) \Rightarrow b)$ is a tautology, i.e. $v^*((a \cap b) \Rightarrow b) = T$ for any v and we do not need to consider it anymore.

2. Formula $\neg a = T$ (we use shorthand notation) if and only if $a = F$ so we get that v must be such that $v(a) = F$

2. We want $(a \cup b) = T$ but v is such that $v(a) = F$ so $(a \cup b) = F \cup b = T)$ if and only if $b = T$

This **means** that for any $v : VAR \longrightarrow \{T, F\}$ such that $v(a) = F, \ v(b) = T$

$$v \models \mathcal{G}$$

and we **proved** that $\mathcal{G}$ is **consistent**

Independent: Example

Example

Show that a formula $A = ((a \Rightarrow b) \cap c)$ is **independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Solution

We construct $v_1,\ v_2 : VAR \longrightarrow \{T, F\}$ such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

We have just proved that any $v : VAR \longrightarrow \{T, F\}$ such that $v(a) = F,\ v(b) = T$ is a **model** for $\mathcal{G}$

Independent: Example

Take as v_1 any truth assignment such that

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T, \quad v_1(c) = T$

We evaluate $v_1{}^*(A) = v_1{}^*((a \Rightarrow b) \cap c) = (F \Rightarrow T) \cap T = T$

This proves that $v_1 \models \mathcal{G} \cup \{A\}$

Take as v_2 any truth assignment such that

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T, \quad v_2(c) = F$

We evaluate $v_2{}^*(\neg A) = v_2{}^*(\neg(a \Rightarrow b) \cap c)) = T \cap T = T$

This proves that $v_2 \models \mathcal{G} \cup \{\neg A\}$

It ends the proof that A is **independent** of $\mathcal{G}$

Not Independent: Example

Example

Show that a formula $A = (\neg a \cap b)$ is **not independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Solution

We have to show that **it is impossible** to construct v_1, v_2 such that

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

Observe that we have just proved that any v such that $v(a) = F,$ and $v(b) = T$ is the only model restricted to the set of variables $\{a, b\}$ for $\mathcal{G}$ and $\{a, b\} = VAR_A$

So we have to check now if it is **possible** $v \models A$ and $v \models \neg A$

Not Independent: Example

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for

$v(a) = F$, and $v(b) = T$

$v^*(A) = v^*((\neg a \cap b) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$

and so $v \models A$

$v^*(\neg A) = \neg v^*(A) = \neg T = F$

and so $v \not\models \neg A$

This end the proof that A is **not independent** of $\mathcal{G}$

Independent: Another Example

Example

Given a set $\mathcal{G} = \{a, (a \Rightarrow b)\}$, **find** a formula A that is **independent** from $\mathcal{G}$

Observe that v such that $v(a) = T, v(b) = T$ is **the only** restricted model for $\mathcal{G}$

So we have to come up with a formula A such that there are two different truth assignments, v_1 and v_2, and

$$v_1 \models \mathcal{G} \cup \{A\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg A\}$$

Let's consider $A = c$, then $\mathcal{G} \cup \{A\} = \{a, (a \Rightarrow b), c\}$

A truth assignment v_1, such that $v_1(a) = T$, $v_1(b) = T$ and $v_1(c) = T$ is a **model** for $\mathcal{G} \cup \{A\}$

Likewise for $\mathcal{G} \cup \{\neg A\} = \{a, (a \Rightarrow b), \neg c\}$

Any v_2, such that $v_2(a) = T$, $v_2(b) = T$ and $v_2(c) = F$ is a **model** for $\mathcal{G} \cup \{\neg A\}$ and so the formula A is **independent**

Challenge Problem

Challenge Problem

Find an infinite number of formulas that are **independent** of a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b),\ (a \cup b), \neg a\}$$

Challenge Problem Solution

This my solution - there are many others- this one seemed to me the **most simple**

Solution

We just proved that any v such that $v(a) = F, \ \ v(b) = T$ is **the only** model restricted to the set of variables $\{a, b\}$ and so all other possible models for $\mathcal{G}$ must be **extensions** of v

Challenge Problem Solution

We **define** a countably infinite set of formulas (and their negations) and corresponding **extensions** of v (restricted to to the set of variables $\{a, b\}$) such that $v \models \mathcal{G}$ as follows

Observe that **all extensions** of v restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$VAR = \{a_1,\ a_2,\ \ldots, a_n \ldots\}$$

We take as an infinite set of formulas in which every formula **independent** of $\mathcal{G}$ the set of **atomic formulas**

$$\mathcal{F}_0 = \{a_1,\ a_2,\ \ldots, a_n \ldots\} - \{a,\ b\}$$

Challenge Problem Solution

Let $c \in \mathcal{F}_0 = \{a_1,\ a_2,\ \ldots, a_n \ldots\} - \{a,\ b\}$

We define truth assignments $v_1,\ v_2 : VAR \longrightarrow \{T, F\}$ such that

$$v_1 \models \mathcal{G} \cup \{c\} \text{ and } v_2 \models \mathcal{G} \cup \{\neg c\}$$

as follows

$v_1(a) = v(a) = F, \quad v_1(b) = v(b) = T$ and $v_1(c) = T$ for any $c \in \mathcal{F}_0$

$v_2(a) = v(a) = F, \quad v_2(b) = v(b) = T$ and $v_2(c) = F$ for any $c \in \mathcal{F}_0$