

cse371/mat371
LOGIC

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LECTURE 3

Chapter 3

Propositional Languages

PART 1: Propositional Languages: **Intuitive Introduction**

PART 2: Propositional Languages: **Formal Definitions**

PART 1: Propositional Languages Intuitive Introduction

We define now a **general notion** of a propositional language.
We show how to obtain, as specific cases, **various languages** for propositional **classical logic** and some **non-classical logics**

We assume the following:

All propositional languages contain an **infinitely countable set of variables** VAR , which elements are denoted by

a, b, c, \dots

with indices, if necessary

All propositional languages share the general way their **sets of formulas** are formed

Propositional Languages

We distinguish one propositional language from the other is the choice of its set of propositional connectives.

We adopt a notation

$$\mathcal{L}_{CON}$$

where CON stands for the set of connectives

We use a notation

$$\mathcal{L}$$

when the set of connectives is fixed

Propositional Languages

For example, the language

$$\mathcal{L}_{\{\neg\}}$$

denotes a propositional language with only one connective \neg
The language

$$\mathcal{L}_{\{\neg, \Rightarrow\}}$$

denotes that a language with two connectives \neg and \Rightarrow
adopted as propositional connectives

Remember: formal languages deal with symbols only and
are also called **symbolic languages**

General Principles

Symbols for connectives do have **intuitive meaning**.

Semantics provides a **formal meaning** of the connectives and is defined separately.

One language can have **many semantics**.

Different logics can share the same language.

For example: the language

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

is used as a propositional language of **classical** and **intuitionistic** logics, some **many-valued** logics, and we **extend** it to the language of many **modal** logics

General Principles

Several languages can share the same semantics.

The classical propositional logic is the best example of such situation.

Due to the **functional dependency** of **classical logic connectives** the languages:

$$\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{L}_{\{\neg, \cap\}}, \mathcal{L}_{\{\neg, \cup\}}, \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}},$$

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow, \Leftrightarrow\}}, \mathcal{L}_{\{\uparrow\}}, \mathcal{L}_{\{\downarrow\}}$$

are all **equivalent** under the **classical semantics**

We will define formally languages **equivalency of languages** in the next chapter.

General Principles

Propositional connectives have well established **names** and the way we read them, even if their **semantics may differ**

We use names **negation, conjunction, disjunction** and **implication** for $\neg, \cap, \cup, \Rightarrow$, respectively

The connective \uparrow is called **alternative negation** and

$A \uparrow B$ reads: **not both A and B**

The connective \downarrow is called **joint negation**

and $A \downarrow B$ reads: **neither A nor B**

Some Non-Classical Propositional Connectives

Other most common **propositional connectives** are **modal** connectives of **possibility** and **necessity**

Modal connectives are not extensional

Standard modal symbols are: \Box for **necessity** and \Diamond for **possibility**.

We will also use symbols **C** and **I** for modal connectives of possibility and necessity, respectively.

The formula $\Diamond A$, or $\Diamond A$ reads: **it is possible that A** or **A is possible** and

The formula $\Box A$, or $\Box A$ reads: **it is necessary that A** or **A is necessary**

Modal Propositional Connectives

Symbols **C** and **I** are used for their **topological** meaning in the semantics of standard **modal logics S4** and **S5**

In topology **C** is a symbol for a set **closure** operation

CA means a **closure** of a set **A**

I is a symbol for a set **interior** operation

IA denotes an **interior** of the set **A**

Modal logics extend the **classical logic**

A modal logic **languages** are for example

$$\mathcal{L}\{C, I, \neg, \cap, \cup, \Rightarrow\} \quad \text{OR} \quad \mathcal{L}\{\Box, \Diamond, \neg, \cap, \cup, \Rightarrow\}$$

Some More Non-Extensional Connectives

Knowledge logics also **extend** the classical logic by adding a new one argument **knowledge** connective

The **knowledge** connective is often denoted by **K**

A formula **KA** reads: **it is known that A** or **A is known**

A language of a **knowledge logic** is for example

$$\mathcal{L}\{K, \neg, \wedge, \vee, \Rightarrow\}$$

Some More Non-Extensional Connectives

Autoepistemic logics extend classical logic by adding an one argument **believe connective**, often denoted by **B**

A formula **BA** reads: **it is believed that A**

A language of an **autoepistemic logic** is for example

$$\mathcal{L}\{B, \neg, \wedge, \vee, \Rightarrow\}$$

Some More Non-Extensional Connectives

Temporal logics also **extend** classical logic by adding one argument **temporal connectives**

Some of temporal connectives are: **F, P, G, H**.

Their **intuitive** meanings are:

FA reads **A is true at some future time**,

PA reads **A was true at some past time**,

GA reads **A will be true at all future times**,

HA reads **A has always been true in the past**

Propositional Connectives

It is possible to create connectives with **more than one or two arguments**

We consider here only **one** or **two argument connectives**

Chapter 3
Propositional Languages
PART 2: Formal Definitions

Propositional Language

Definition

A **propositional language** is a pair

$$\mathcal{L} = (\mathcal{A}, \mathcal{F})$$

where \mathcal{A}, \mathcal{F} are called an **alphabet** and a **set of formulas**, respectively

Definition

Alphabet is a set

$$\mathcal{A} = \text{VAR} \cup \text{CON} \cup \text{PAR}$$

VAR, CON, PAR are all **disjoint** sets of propositional **variables, connectives** and **parenthesis**, respectively

The sets **VAR, CON** are **non-empty**

Alphabet Components

VAR is a **countably infinite** set of **propositional variables**

We denote elements of **VAR** by

a, b, c, d, ...

with indices if necessary

CON $\neq \emptyset$ is a **finite set** of **propositional connectives**

We assume that the set **CON** of logical connectives is **non-empty**, i.e. that a propositional language always has at **least one** connective.

Alphabet Components

Notation

We **denote** the language \mathcal{L} with the set of connectives CON by

$$\mathcal{L}_{CON}$$

Observe that **propositional languages differ** only on a **choice** of the **propositional connectives**, hence our notation.

Alphabet Components

PAR is a set of **auxiliary symbols**

This set **may be empty**; for example in case of Polish notation.

Assumptions

We assume here that **PAR** contains only 2 **parenthesis** and

$$PAR = \{ (,) \}$$

We also assume that the set **CON** of **logical connectives** contains only **unary** and **binary** connectives, i.e.

$$CON = C_1 \cup C_2$$

where C_1 is the set of all **unary** connectives, and C_2 is the set of all **binary** connectives

Formulas Definition

Definition

The set \mathcal{F} of **all formulas** of a propositional language \mathcal{L}_{CON} is build **recursively** from the elements of the alphabet \mathcal{A} as follows.

$\mathcal{F} \subseteq \mathcal{A}^*$ and \mathcal{F} is the **smallest** set for which the following conditions are satisfied

- (1) $VAR \subseteq \mathcal{F}$
- (2) If $A \in \mathcal{F}$, $\nabla \in C_1$, then $\nabla A \in \mathcal{F}$
- (3) If $A, B \in \mathcal{F}$, $\circ \in C_2$ i.e \circ is a two argument connective, then $(A \circ B) \in \mathcal{F}$

By (1) **propositional variables** are formulas and they are called **atomic formulas**

The set \mathcal{F} is also called a set of all **well formed formulas** (wff) of the language \mathcal{L}_{CON}

Set of Formulas

Observe that the the alphabet \mathcal{A} is **countably infinite**

Hence the set \mathcal{A}^* of all finite sequences of elements of \mathcal{A} is also **countably infinite**

By definition $\mathcal{F} \subseteq \mathcal{A}^*$ and hence we get that the set of all formulas \mathcal{F} is also **countably infinite**

We state as separate fact

Fact

For any propositional language $\mathcal{L} = (\mathcal{A}, \mathcal{F})$, its sets of formulas \mathcal{F} is always a **countably infinite** set

We hence consider here only **infinitely countable languages**

Main Connectives and Direct Sub-Formulas

∇ is called a **main connective** of the formula $\nabla A \in \mathcal{F}$

A is called its **direct sub-formula** of ∇A

\circ is called a **main connective** of the formula $(A \circ B) \in \mathcal{F}$

A, B are called **direct sub-formulas** of $(A \circ B)$

Examples

E1 Main connective of $(a \Rightarrow \neg Nb)$ is \Rightarrow
 $a, \neg Nb$ are direct sub-formulas

E2 Main connective of $N(a \Rightarrow \neg b)$ is N
 $(a \Rightarrow \neg b)$ is the direct sub-formula

E3 Main connective of $\neg(a \Rightarrow \neg b)$ is \neg
 $(a \Rightarrow \neg b)$ is the direct sub-formula

E4 Main connective of $(\neg a \cup \neg(a \Rightarrow b))$ is \cup
 $\neg a, \neg(a \Rightarrow b)$ are direct sub-formulas

Sub-Formulas

We define a notion of a **sub-formula** in two steps:

Step 1

For any formulas A and B , the formula A is a **proper sub-formula** of B if there is sequence of formulas, beginning with A , ending with B , and in which each term is a **direct sub-formula** of the next

Step 2

A **sub-formula** of a given formula A is any **proper sub-formula** of A , or A itself

Sub-Formulas Example

The formula $(\neg a \cup \neg(a \Rightarrow b))$

has two **direct sub-formulas**: $\neg a$, $\neg(a \Rightarrow b)$

The **direct sub-formulas** of $\neg a$, $\neg(a \Rightarrow b)$
are respectively a , $(a \Rightarrow b)$

The direct sub-formulas of a , $(a \Rightarrow b)$, are a , b

END of the process

Example

Given a formula

$$(\neg a \cup \neg(a \Rightarrow b))$$

Its set of all **proper sub-formulas** is:

$$S = \{\neg a, \neg(a \Rightarrow b), a, (a \Rightarrow b), b\}$$

The set of all its **sub-formulas** is

$$S \cup \{(\neg a \cup \neg(a \Rightarrow b))\}$$

Formula Degree Definition

We **define** a **degree of a formula** as a **number** of occurrences of logical connectives in the formula.

Example

The **degree** of $(\neg a \cup \neg(a \Rightarrow b))$ is **4**

The **degree** of $\neg(a \Rightarrow b)$ is **2**

The **degree** of $\neg a$ is **1**

The **degree** of a is **0**

Formula Degree

A degree of a formula is number of occurrences of logical connectives in the formula

Observation: the degree of any proper sub-formula of A must be one less than the degree of A

This is the central fact upon which mathematical induction arguments are based.

Proofs of properties of formulas are usually carried by mathematical induction on their degrees

Exercise

Exercise 1

Consider a language

$$\mathcal{L} = \mathcal{L}_{\{\neg, \diamond, \square, \cup, \cap, \Rightarrow\}}$$

and a set $S \subseteq \mathcal{A}^*$ such that

$$S = \{\diamond\neg a \Rightarrow (a \cup b), (\diamond(\neg a \Rightarrow (a \cup b))), \\ \diamond\neg(a \Rightarrow (a \cup b))\}$$

1. **Determine** which of the elements of S are, and which are not **well formed formulas (wff)** of \mathcal{L}
2. If a formula A is a **well formed formula**, i.e. $A \in \mathcal{F}$, determine its **main connective**.
3. If $A \notin \mathcal{F}$ write the correct formula and then determine its **main connective**

Exercise 1 Solution

Solution

The formula $\diamond\neg a \Rightarrow (a \cup b)$ **is not a well formed formula**

The **correct** formula is

$$(\diamond\neg a \Rightarrow (a \cup b))$$

The **main connective** is \Rightarrow

The **correct** formula says:

If negation of a is possible, then we have a or b

Another correct formula in is

$$\diamond(\neg a \Rightarrow (a \cup b))$$

The main connective is \diamond

The corrected formula says:

It is possible that not a implies a or b

Exercise 1 Solution

The formula $(\diamond(\neg a \Rightarrow (a \cup b)))$ **is not correct**

The **correct** formula is

$$\diamond(\neg a \Rightarrow (a \cup b))$$

The **main connective** is \diamond

The **correct** formula says:

It is possible that not a implies a or b

$\diamond\neg(a \Rightarrow (a \cup b))$ is a **correct formula**

The main connective is \diamond

The formula says:

It is possible that it is not true that a implies a or b

Exercise

Exercise 2

Given a formula:

$$\diamond((a \cup \neg a) \cap b)$$

1. Determine its **degree**
2. Write down all its **sub-formulas**

Solution:

The degree is **4**

All **sub-formulas** are:

$$\diamond((a \cup \neg a) \cap b), ((a \cup \neg a) \cap b),$$

$$(a \cup \neg a), \neg a, b, a$$

Language Defined by a set S

Definition

Given a set S of formulas of a language \mathcal{L}_{CON}

Let $CS \subseteq CON$ be the set of **all connectives** that appear in formulas of S

A language \mathcal{L}_{CS}

is called the **language defined** by the set of formulas S

Example

Let S be a set

$$S = \{((a \Rightarrow \neg b) \Rightarrow \neg a), \Box(\neg \Diamond a \Rightarrow \neg a)\}$$

All connectives appearing in the formulas in S are:

$$\Rightarrow, \neg, \Box, \Diamond$$

The **language defined** by the set S is

$$\mathcal{L}_{\{\neg, \Rightarrow, \Box, \Diamond\}}$$

Exercise

Exercise 3

Write the following natural language statement:

From the fact that it is possible that Anne is not a boy we deduce that it is not possible that Anne is not a boy or, if it is possible that Anne is not a boy, then it is not necessary that Anne is pretty

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Exercise 3 Solution

1. We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of the language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *Anne is a boy*,

b denotes a statement: *Anne is pretty*

Propositional Modal Connectives: \Box, \Diamond

\Diamond denotes statement: *it is possible that*

\Box denotes statement: *it is necessary that*

Translation 1: the formula A_1 is

$$(\Diamond \neg a \Rightarrow (\neg \Diamond \neg a \cup (\Diamond \neg a \Rightarrow \neg \Box b)))$$

Exercise 3 Solution

2. We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of the language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *it is possible that Anne is not a boy*

b denotes a statement: *it is necessary that Anne is pretty*

Translation 2: the formula A_2 is

$$(a \Rightarrow (\neg a \cup (a \Rightarrow \neg b)))$$

Exercise

Exercise 4

Write the following natural language statement:

For all natural numbers $n \in \mathbb{N}$ the following implication holds: if $n < 0$, then there is a natural number m , such that it is possible that $n + m < 0$, OR it is not possible that there is a natural number m , such that $m > 0$

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$

Exercise 4 Solution

1. We translate our statement into a formula $A_1 \in \mathcal{F}_1$ of the language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *For all natural numbers $n \in \mathbb{N}$ the following implication holds: if $n < 0$, then there is a natural number m , such that it is possible that $n + m < 0$*

b denotes a statement: *it is not possible that there is a natural number m , such that $m > 0$*

Translation: the formula A_1 is

$$(a \cup \neg b)$$

Exercise 4 Solution

2. We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *For all natural numbers $n \in \mathbb{N}$ the following implication holds: if $n < 0$, then there is a natural number m , such that it is possible that $n + m < 0$*

b denotes a statement: *there is a natural number m , such that $m > 0$*

Translation: the formula A_2 is

$$(a \cup \neg \diamond b)$$

Exercise

Exercise 5

Write the following natural language statement:

*The following statement holds for all natural numbers $n \in \mathbb{N}$:
if $n < 0$, then there is a natural number m , such that it is
possible that $n + m < 0$, OR it is not possible that there is a
natural number m , such that $m > 0$*

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$

Exercise

Exercise 6

Write the following natural language statement:

From the fact that each natural number is greater than zero we deduce that it is not possible that Anne is a boy or, if it is possible that Anne is not a boy, then it is necessary that it is not true that each natural number is greater than zero

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$