LECTURE 10a
Chapter 10
Predicate Automated Proof Systems

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Part 3: Proof of Completeness Theorem for QRS
Proof of Completeness Theorem

The proof of completeness theorem presented here is due to Rasiowa and Sikorski (1961), as is the proof system QRS. We adopted their proof to propositional case in chapter 6.

The completeness proofs, in the propositional case and in predicate case, are based on a direct construction of a counter model for any unprovable formula.

The construction of the counter model for the unprovable formula $A$ uses the decomposition tree $T_A$.

We call such constructed counter model a counter model determined by the tree $T_A$. 
Proof of Completeness Theorem

Given a first order language $\mathcal{L}$ with the set $\text{VAR}$ of variables and the set $\mathcal{F}$ of formulas.

We define, after chapter 8 a notion of a model and a counter-model of a formula $A$ of $\mathcal{L}$ and then extend it to the the set $\mathcal{F}^*$ establishing the semantics for QRS.
Proof of Completeness Theorem

Model
A structure \( \mathcal{M} = [M, I] \) is called a model of \( A \in \mathcal{F} \) if and only if
\[
(\mathcal{M}, \nu) \models A
\]
for all assignments \( \nu : \text{VAR} \rightarrow M \).
We denote it by
\[
\mathcal{M} \models A
\]

\( M \) is called the universe of the model, \( I \) the interpretation.
Proof of Completeness Theorem

Counter - Model
A structure $\mathcal{M} = [M, I]$ is called a counter-model of $A \in \mathcal{F}$ if and only if there is $v : \text{VAR} \rightarrow M$, such that

$$(\mathcal{M}, v) \not\models A$$

We denote it by

$\mathcal{M} \not\models A$
Proof of Completeness Theorem

Tautology
A formula $A \in \mathcal{F}$ is called a predicate tautology and denoted by $\models A$ if and only if all structures $M = [M, I]$ are models of $A$, i.e.

$$\models A \text{ if and only if } M \models A$$

for all structures $M = [M, I]$ for $\mathcal{L}$
Proof of Completeness Theorem

For any sequence \( \Gamma \in \mathcal{F}^* \), by

\[ \delta_{\Gamma} \]

we understand any disjunction of all formulas of \( \Gamma \). A structure \( \mathcal{M} = [M, I] \) is called a model of a \( \Gamma \in \mathcal{F}^* \) and denoted by

\[ \mathcal{M} \models \Gamma \]

if and only if

\[ \mathcal{M} \models \delta_{\Gamma} \]

The sequence \( \Gamma \) is a predicate tautology if and only if the formula \( \delta_{\Gamma} \) is a predicate tautology, i.e.

\[ \models \Gamma \text{ if and only if } \models \delta_{\Gamma} \]
Proof of Completeness Theorem

Completeness Theorem
For any $\Gamma \in F^*$,

$$\vdash_{QRS} \Gamma \text{ if and only if } \models \Gamma$$

In particular, for any formula $A \in F$,

$$\vdash_{QRS} A \text{ if and only if } \models A$$

Proof  We prove the completeness part. We need to prove the formula $A$ case only because the case of a sequence $\Gamma$ can be reduced to the formula case of $\delta_\Gamma$. I.e. we prove the implication:

$$\text{if } \models A, \text{ then } \vdash_{QRS} A$$
Proof of Completeness Theorem

We do it, as in the propositional case, by proving the opposite implication

$$\text{if } \kappa_{QRS} A \text{ then } \not\models A$$

This means that we want to prove that for any formula $A$, unprovability of $A$ in QRS allows us to define its countermodel.

The counter-model is determined, as in the propositional case, by the decomposition tree $T_A$

We have proved the following

**Tree Theorem**

Each formula $A$, generates its unique decomposition tree $T_A$ and $A$ has a proof only if this tree is finite and all its end sequences (leaves) are axioms.
Proof of Completeness Theorem

The Tree Theorem says that we have two cases to consider:

(C1) the tree $T_A$ is finite and contains a leaf which is not axiom, or

(C2) the tree $T_A$ is infinite

We will show how to construct a counter-model for $A$ in both cases:

a counter-model determined by a non-axiom leaf of the decomposition tree $T_A$,

or a counter-model determined by an infinite branch of $T_A$
Proof of Completeness Theorem

**Proof in case (C1)**
The tree $T_A$ is **finite** and contains a non-axiom leaf
Before describing a **general method** of constructing the counter-model determined by the decomposition tree $T_A$ we describe it, as an example, for a case of a general formula

$$(\exists x A(x) \Rightarrow \forall x A(x)),$$

and its **particular case**

$$(\exists x (P(x) \cap R(x, y)) \Rightarrow \forall x (P(x) \cap R(x, y))),$$

where $P, R$ are one and two argument predicate symbols, respectively.
Proof of Completeness Theorem

First we build its decomposition tree:

\[ T_A \]

\[(\exists x (P(x) \cap R(x, y)) \Rightarrow \forall x (P(x) \cap R(x, y)))\]

| (\Rightarrow)
\[-\exists x (P(x) \cap R(x, y)), \forall x (P(x) \cap R(x, y))\]

| (\neg \exists)
\[\forall x \neg (P(x) \cap R(x, y)), \forall x (P(x) \cap R(x, y))\]

| (\forall)
\[-(P(x_1) \cap R(x_1, y)), \forall x (P(x) \cap R(x, y))\]

where \(x_1\) is a first free variable in the sequence of term ST such that \(x_1\) does not appear in \(\forall x \neg (P(x) \cap R(x, y)), \forall x (P(x) \cap R(x, y))\)

| (\neg \cap)
\[-P(x_1), \neg R(x_1, y), \forall x (P(x) \cap R(x, y))\]

| (\forall)
Proof of Completeness Theorem

\[ \neg P(x_1), \neg R(x_1, y), (P(x_2) \cap R(x_2, y)) \]

where \( x_2 \) is a first free variable in the sequence of term ST such that \( x_2 \) does not appear in 
\( \neg P(x_1), \neg R(x_1, y), \forall x (P(x) \cap R(x, y)) \), the sequence ST is one-to-one, hence \( x_1 \neq x_2 \)

\[ \land (\cap) \]

\[ \neg P(x_1), \neg R(x_1, y), P(x_2) \quad \neg P(x_1), \neg R(x_1, y), R(x_2, y) \]

\( x_1 \neq x_2 \), Non-axiom

\( x_1 \neq x_2 \), Non-axiom
Proof of Completeness Theorem

There are two non-axiom leaves

In order to define a counter-model determined by the tree $T_A$ we need to chose only one of them

Let's choose the leaf

$L_A = \neg P(x_1), \neg R(x_1, y), P(x_2)$

We use the non-axiom leaf $L_A$ to define a structure $\mathcal{M} = [M, I]$ and an assignment $\nu$, such that

$(\mathcal{M}, \nu) \not\models A$

Such defined $\mathcal{M}$ is called a counter-model determined by the tree $T_A$
Proof of Completeness Theorem

We take a the **universe** of $\mathcal{M}$ the set $\mathbf{T}$ of **all terms** of our language $\mathcal{L}$, i.e. we put $\mathcal{M} = \mathbf{T}$.

We define the **interpretation** $I$ as follows.

For any **predicate symbol** $Q \in \mathbf{P}, \#Q = n$ we put that $Q_I(t_1, \ldots t_n)$ is **true** (holds) for terms $t_1, \ldots t_n$ if and only if the negation $\neg Q_I(t_1, \ldots t_n)$ of the formula $Q(t_1, \ldots t_n)$ appears on the leaf $L_A$ and $Q_I(t_1, \ldots t_n)$ is **false** (does not hold) for terms $t_1, \ldots t_n$, otherwise.

For any **functional symbol** $f \in \mathbf{F}, \#f = n$ we put

$$f_I(t_1, \ldots t_n) = f(t_1, \ldots t_n)$$
Proof of Completeness Theorem

It is easy to see that in particular case of our non-axiom leaf

\[ L_A = \neg P(x_1), \neg R(x_1, y), P(x_2) \]

\( P_I(x_1) \) is true for \( x_1 \), and not true for \( x_2 \)

\( R_I(x_1, y) \) is true (holds) holds for \( x_1 \) and for any \( y \in VAR \)
Proof of Completeness Theorem

We define the assignment \( \nu : \text{VAR} \rightarrow T \) as identity, i.e., we put \( \nu(x) = x \) for any \( x \in \text{VAR} \).

Obviously, for such defined structure \([M, I]\) and the assignment \( \nu \) we have that

\[
([T, I], \nu) \models P(x_1), ([T, I], \nu) \models R(x_1, y) \quad \text{and} \quad ([T, I], \nu) \not\models P(x_2)
\]

We hence obtain that

\[
([T, I], \nu) \not\models \neg P(x_1), \neg R(x_1, y), P(x_2)
\]

This proves that such defined structure \([T, I]\) is a counter model for a non-axiom leaf \( L_A \) and that \( A \) is not tautology.