cse371/mat371
LOGIC

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LECTURE 10a

Chapter 10
Predicate Automated Proof Systems

Part 1: Predicate Languages

Part 2: Proof System **QRS**

Part 3: Proof of Completeness Theorem for **QRS**

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Part 3: Proof of Completeness Theorem for **QRS**

Proof of Completeness Theorem

The proof of completeness theorem presented here is due to Rasiowa and Sikorski (1961), as is the proof system **QRS**.

We adopted their proof to propositional case in chapter 6

The completeness proofs, in the propositional case and in predicate case, are based on a **direct construction** of a counter model for any unprovable formula.

The construction of the counter model for the unprovable formula A uses the decomposition tree $\mathbf{T}_A$

We call such constructed counter model a **counter model determined** by the tree $\mathbf{T}_A$

Proof of Completeness Theorem

Given a first order language $\mathcal{L}$ with the set VAR of variables and the set $\mathcal{F}$ of formulas

We define, after chapter 8 a notion of a **model** and a **counter-model** of a formula A of $\mathcal{L}$ and then **extend** it to the the set $\mathcal{F}^*$ establishing the **semantics** for **QRS**

Proof of Completeness Theorem

Model

A structure $\mathcal{M} = [M, I]$ is called a **model** of $A \in \mathcal{F}$

if and only if

$$(\mathcal{M}, v) \models A$$

for all assignments $v : VAR \longrightarrow M$

We denote it by

$$\mathcal{M} \models A$$

M is called the **universe** of the model, I the interpretation

Proof of Completeness Theorem

Counter - Model

A structure $\mathcal{M} = [M, I]$ is called a **counter- model** of $A \in \mathcal{F}$ if and only if **there is** $v : VAR \longrightarrow M$, such that

$$(\mathcal{M}, v) \not\models A$$

We denote it by

$$\mathcal{M} \not\models A$$

Proof of Completeness Theorem

Tautology

A formula $A \in \mathcal{F}$ is called a **predicate tautology** and denoted by $\models A$ if and only if

all structures $\mathcal{M} = [M, I]$ are **models** of A, i.e.

$$\models A \quad \text{if and only if} \quad \mathcal{M} \models A$$

for all structures $\mathcal{M} = [M, I]$ for $\mathcal{L}$

Proof of Completeness Theorem

For any sequence $\Gamma \in \mathcal{F}^*$, by

$$\delta_\Gamma$$

we understand any **disjunction** of all formulas of Γ

A structure $\mathcal{M} = [M, I]$ is called a **model** of a $\Gamma \in \mathcal{F}^*$ and denoted by

$$\mathcal{M} \models \Gamma$$

if and only if

$$\mathcal{M} \models \delta_\Gamma$$

The sequence Γ is a **predicate tautology** if and only if the formula δ_Γ is a predicate tautology, i.e.

$$\models \Gamma \quad \text{if and only if} \quad \models \delta_\Gamma$$

Proof of Completeness Theorem

Completeness Theorem

For any $\Gamma \in \mathcal{F}^*$,

$$\vdash_{QRS} \Gamma \text{ if and only if } \models \Gamma$$

In particular, for any formula $A \in \mathcal{F}$,

$$\vdash_{QRS} A \text{ if and only if } \models A$$

Proof We prove the completeness part. We need to prove the formula A case only because the case of a sequence Γ can be reduced to the formula case of δ_Γ. I.e. we prove the implication:

$$\text{if } \models A, \text{ then } \vdash_{QRS} A$$

Proof of Completeness Theorem

We do it, as in the propositional case, by proving the opposite implication

$$\text{if} \quad \nvdash_{QRS} \ A \quad \text{then} \quad \not\models \ A$$

This means that we want prove that for any formula A, **unprovability** of A in **QRS** allows us to define its **counter-model**.

The counter- model is determined, as in the propositional case, by the decomposition tree $\mathbf{T}_A$

We have proved the following

Tree Theorem

Each formula A, generates its unique decomposition tree $\mathcal{T}_A$ and A has a proof only if this tree is finite and all its end sequences (leaves) are axioms.

Proof of Completeness Theorem

The **Tree Theorem** says says that we have two cases to consider:

(C1) the tree $\mathbf{T}_A$ is **finite** and contains a leaf which is not axiom, or

(C2) the tree $\mathbf{T}_A$ is **infinite**

We will show how to construct a counter- model for A in both cases:

a counter- model determined by a non-axiom leaf of the decomposition tree $\mathbf{T}_A$,

or a counter- model determined by an infinite branch of $\mathbf{T}_A$

Proof of Completeness Theorem

Proof in case **(C1)**

The tree $\mathbf{T}_A$ is **finite** and contains a non- axiom leaf

Before describing a general method of constructing the counter-model determined by the decomposition tree $\mathcal{T}_A$ we describe it, as an example, for a case of a general formula

$$(\exists x A(x) \Rightarrow \forall x A(x)),$$

and its **particular case**

$$(\exists x(P(x) \cap R(x,y)) \Rightarrow \forall x(P(x) \cap R(x,y))),$$

where P, R are one and two argument predicate symbols, respectively.

Proof of Completeness Theorem

First we build its decomposition tree:

$$\mathbf{T}_A$$

$$(\exists x(P(x) \cap R(x,y)) \Rightarrow \forall x(P(x) \cap R(x,y)))$$

$$\mid (\Rightarrow)$$

$$\neg \exists x(P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\neg \exists)$$

$$\forall x \neg (P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

$$\neg (P(x_1) \cap R(x_1,y)), \forall x(P(x) \cap R(x,y))$$

where x_1 is a first free variable in the sequence of term ST such that x_1 does not appear in $\forall x \neg (P(x) \cap R(x,y)), \forall x(P(x) \cap R(x,y))$

$$\mid (\neg \cap)$$

$$\neg P(x_1), \neg R(x_1,y), \forall x(P(x) \cap R(x,y))$$

$$\mid (\forall)$$

Proof of Completeness Theorem

$$\mid (\forall)$$

$$\neg P(x_1), \neg R(x_1, y), (P(x_2) \cap R(x_2, y))$$

where x_2 is a first free variable in the sequence of term ST such that x_2 does not appear in $\neg P(x_1), \neg R(x_1, y), \forall x(P(x) \cap R(x, y))$, the sequence ST is one-to- one, hence $x_1 \neq x_2$

$$\bigwedge (\cap)$$

$$\neg P(x_1), \neg R(x_1, y), P(x_2)$$

$x_1 \neq x_2$, Non-axiom

$$\neg P(x_1), \neg R(x_1, y), R(x_2, y)$$

$x_1 \neq x_2$, Non-axiom

Proof of Completeness Theorem

There are two non-axiom leaves

In order to define a counter-model determined by the tree $\mathbf{T}_A$ we need to chose only one of them

Let's choose the leaf

$$L_A = \neg P(x_1), \neg R(x_1, y), P(x_2)$$

We use the **non-axiom leaf** L_A to define a structure $\mathcal{M} = [M, I]$ and an assignment v, such that

$$(\mathcal{M}, v) \not\models A$$

Such defined $\mathcal{M}$ is called a **counter - model** determined by the tree $\mathbf{T}_A$

Proof of Completeness Theorem

We take a the **universe** of $\mathcal{M}$ the set **T** of all terms of our language $\mathcal{L}$, i.e. we put $M = \mathbf{T}$.

We define the **interpretation** I as follows.

For any **predicate symbol** $Q \in \mathbf{P}, \#Q = n$ we put that

$Q_I(t_1, \ldots t_n)$ is **true** (holds) for terms $t_1, \ldots t_n$

if and only if

the negation $\neg Q_I(t_1, \ldots t_n)$ of the formula $Q(t_1, \ldots t_n)$ **appears** on the leaf L_A

and $Q_I(t_1, \ldots t_n)$ is **false** (does not hold) for terms $t_1, \ldots t_n$, otherwise

For any **functional symbol** $f \in \mathbf{F}, \#f = n$ we put

$$f_I(t_1, \ldots t_n) = f(t_1, \ldots t_n)$$

Proof of Completeness Theorem

It is easy to see that in particular case of our non-axiom leaf

$$L_A = \neg P(x_1), \ \neg R(x_1, y), \ P(x_2)$$

$P_I(x_1)$ is **true** for x_1, and **not true** for x_2

$R_I(x_1, y)$ is **true** (holds) holds fo2 rx_1 and for any $y \in VAR$

Proof of Completeness Theorem

We define the assignment $v : VAR \longrightarrow T$ as **identity**, i.e., we put $v(x) = x$ for any $x \in VAR$

Obviously, for such defined structure $[M, I]$ and the assignment v we have that

$$([\mathbf{T}, I],\ v) \models P(x_1),\ ([\mathbf{T}, I],\ v) \models R(x_1, y) \text{ and } ([\mathbf{T}, I],\ v) \not\models P(x_2)$$

We hence obtain that

$$([\mathbf{T}, I],\ v) \not\models \neg P(x_1), \neg R(x_1, y), P(x_2)$$

This proves that such defined structure $[\mathbf{T}, I]$ is a counter model for a non-axiom leaf L_A and that A is not tautology