Problem 1

Write the following natural language statement:

*From the fact that each natural number is greater than zero we deduce that: it is not possible that Anne is a boy or, if it is possible that Anne is not a boy, then it is necessary that it is not true that each natural number is greater than zero in the following two ways.*

1. As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

Solution

**Propositional Variables:** $a, b$, where

- $a$ denotes statement: *each natural number is greater than zero,*
- $b$ denotes statement: *Anne is a boy*

**Propositional Modal Connectives:** $\Box, \Diamond$

- $\Diamond$ denotes statement: *it is possible that,*
- $\Box$ denotes statement: *it is necessary that*

**Translation** The formula $A_1$ is

$$(a \Rightarrow (\neg \Diamond b \cup (\Diamond \neg b \Rightarrow \Box \neg a)))$$

2. As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Solution

**Propositional Variables:** $a, b, c, d$ where

- $a$ denotes statement: *each natural number is greater than zero,*
- $b$ denotes statement: *possible that Anne is a boy*
- $c$ denotes statement: *possible that Anne is not a boy*
- $d$ denotes statement: *necessary that it is not true that each natural number is greater than zero*

Formula $A_2$ is

$$(a \Rightarrow (\neg b \cup (c \Rightarrow d)))$$

Problem 2

**Circle** formulas that are propositional/predicate tautologies

$S_1 = \{(A \Rightarrow (A \cup B)), \ ((a \Rightarrow b) \cap (a \Rightarrow c)) \Rightarrow (a \Rightarrow b)),\ (A \cup \neg A),\ (A \cup (A \Rightarrow B)),\ (a \cup \neg b)\}$

$S_2 = \{(\forall x A(x) \Rightarrow \exists x A(x)), \ (\forall x P(x,y) \Rightarrow \exists x P(x,y)), \ ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x (A(x) \cap B(x)))\}$

**Solutions**

- $\not\models (a \cup \neg b)$
- $\not\models ((\exists x A(x) \cap \exists x B(x)) \Rightarrow \exists x (A(x) \cap B(x)))$

Problem 3

Given a formula $A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$
and its restricted model \( v_A : \{a, b, c\} \rightarrow \{T, F\} \), \( v_A(a) = T, v_A(b) = T, v_A(c) = F \)

Extend \( v_A \) to the set of all propositional variables \( \text{VAR} \) to obtain 2 different, non restricted models for \( A \)

**Solution**

**Model** \( w_1 \) is a function

\[
\begin{align*}
  w_1 : \text{VAR} \rightarrow \{T, F\} & \quad \text{such that} \\
  w_1(a) &= v_A(a) = T, \quad w_1(b) = v_A(b) = T, \\
  w_1(c) &= v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \quad x \in \text{VAR} - \{a, b, c\}
\end{align*}
\]

**Model** \( w_2 \) is defined by a formula

\[
\begin{align*}
  w_2(a) &= v_A(a) = T, \quad w_2(b) = v_A(b) = T, \\
  w_2(c) &= v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \quad x \in \text{VAR} - \{a, b, c\}
\end{align*}
\]

**Problem 4**

1. Give an example of an infinite set of formulas of \( \mathcal{L}_{\{\neg, \cup\}} \), different from the set \( T \) of tautologies that consistent. JUSTIFY your answer.

**Reminder:** a set \( G \subseteq \mathcal{T} \) of formulas is called consistent if and only if \( G \) has a model

**Solution**

There plenty of examples; here is the simplest one: \( G = \mathcal{VAR} \)

\[
\begin{align*}
  v : \text{VAR} \rightarrow \{T, F\}, \quad \text{such that} \quad v(x) = T \quad \text{for all} \quad x \in \text{VAR} \quad \text{is obviously a model for each formula in} \ G \quad \text{and hence by definition is a model for} \ G.
\end{align*}
\]

MORE examples in chapter 3 and corresponding Lectures.

2. Give an example of an infinite set of formulas of \( \mathcal{L}_{\{\neg, \cup\}} \), different from the set \( C \) of contradictions that is inconsistent.

**Reminder:** a set \( G \subseteq \mathcal{T} \) is called inconsistent if and only if \( G \) does not have a model

**Solution**

There plenty of examples; here is the simplest one:

Let \( c \) be any variable, i.e. \( c \in \text{VAR} \), we take

\[
G = \text{VAR} \cup \{c, \neg c\}
\]

Obviously, the finite set \( \{c, \neg c\} \) does not have a model, and hence the infinite set \( \text{VAR} \cup \{c, \neg c\} \) does not have a model and hence, by definition is inconsistent.

MORE examples in chapter 3 and corresponding Lectures.

**Reminder:** we define \( \mathcal{H} \) semantics operations \( \cup \) and \( \cap \) as \( x \cup y = \max\{x, y\} \), \( x \cap y = \min\{x, y\} \)

the implication and negation are defined as

\[
x \Rightarrow y = \begin{cases} 
  T & \text{if} \ x \leq y \\
  y & \text{otherwise}
\end{cases}
\]

\[\neg x = x \Rightarrow F\]

**Problem 5**
We know that $v : VAR \rightarrow \{ F, \bot, T \}$ is such that $v^*(a \land b \Rightarrow (a \Rightarrow c)) = \bot$ under H semantics.

Evaluate $v^*((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

You can use SHORTHAND notation.

**Solution** Look at Lecture 3b.