cse371/mat371 LOGIC

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LECTURE 9a
PART 1: Introduction

PART 2: System S Definition and Proof of the Main Lemma

PART 3: Proof 1: Constructive Proof of Completeness Theorem

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PART 4
Proof 2: General Proof of Completeness Theorem
Proof 2
A Counter- Model Existence Method

We prove now the **Completeness Theorem** by proving the opposite implication:

\[ \neg \vdash A \implies \not \models A \]

The **proof** consists of defining a method that uses the information that \( A \) is **not provable** in order to define a **counter-model** for \( A \). We hence call it a **counter-model existence method**.
Proof 2 Steps

The construction of a counter-model for any non-provable $A$ presented in this proof is less constructive then in the case of our first proof.

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

It is hence a much more general method then the first one and this is the reason we present it here.
Proof 2 Steps

We remind that $\not\models A$ means that there is a variable truth assignment $\nu : \text{VAR} \rightarrow \{T, F\}$, such that as we are in classical semantics $\nu^*(A) = F$

We assume that $A$ does not have a proof in $S$, i.e. $\not\vdash A$ we use this information in order to define a general method of constructing $\nu$, such that $\nu^*(A) = F$

This is done in the following steps.
Proof 2 Steps

Step 1
Definition of a special set of formulas $\Delta^*$
We use the information $\not\vdash A$ to define a set of formulas $\Delta^*$ such that $\neg A \in \Delta^*$

Step 2
Definition of the counter-model
We define the variable truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} 
T & \text{if } \Delta^* \vdash a \\
F & \text{if } \Delta^* \vdash \neg a 
\end{cases}$$
Proof 2 Steps

Step 3
We prove that \( v \) is a **counter-model** for \( A \)
We first prove a following more general property of \( v \)

**Property**
The set \( \Delta^* \) and \( v \) defined in the Steps 1 and 2 are such that for every formula \( B \in \mathcal{F} \)

\[
v^*(B) = \begin{cases} 
T & \text{if } \Delta^* \vdash B \\
F & \text{if } \Delta^* \vdash \neg B 
\end{cases}
\]

We then use the **Step 3** to prove that \( v^*(A) = F \)
Main Notions

The definition, construction and the properties of the set $\Delta^*$ and hence the Step 1, are the most essential for the proof.

The other steps have mainly technical character.

The main notions involved in the proof are: consistent set, complete set and a consistent complete extension of a set of formulas.

We are going to prove some essential facts about them.
Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical

Semantical definition uses the notion of a model and says:

A set is **consistent** if it has a model

Syntactical definition uses the notion of provability and says:

A set is **consistent** if one can’t prove a contradiction from it
Consistent and Inconsistent Sets

In our proof of the **Completeness Theorem** we use the following formal **syntactical definition** of consistency of a set of formulas.

**Definition of a consistent set**

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if

there is no a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$
Consistent and Inconsistent Sets

**Definition of an inconsistent set**

A set $\Delta \subseteq F$ is **inconsistent** if and only if there is a formula $A \in F$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

The notion of consistency, as defined above, is characterized by the following **Consistency Lemma**
Consistency Condition Lemma

**Lemma**  Consistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) $\Delta$ is consistent

(ii) there is a formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$
Proof of Consistency Lemma

Proof
To establish the equivalence of (i) and (ii) we prove the corresponding opposite implications.

We prove the following two cases:

Case 1  not (ii)  implies  not (i)

Case 2  not (i)  implies  not (ii)
Proof of Consistency Lemma

Case 1

Assume that not (ii)
It means that for all formulas $A \in \mathcal{F}$ we have that

$$\Delta \vdash A$$

In particular it is true for a certain $A = B$ and for a certain $A = \neg B$ i.e.

$$\Delta \vdash B \quad \text{and} \quad \Delta \vdash \neg B$$

and hence it proves that $\Delta$ is inconsistent
i.e. not (i) holds
Proof of Consistency Lemma

Case 2
Assume that not (i), i.e that $\Delta$ is inconsistent
Then there is a formula $A$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$
Let $B$ be any formula
We assumed (6.) about $S$ that $\vdash (\neg A \Rightarrow (A \Rightarrow B))$
By monotonicity

$$\Delta \vdash (\neg A \Rightarrow (A \Rightarrow B))$$

Applying Modus Ponens twice to $\neg A$ first, and to $A$ next we get that $\Delta \vdash B$ for any formula $B$
Thus not (ii) and it ends the proof of the Lemma
Inconsistency Condition Lemma

Inconsistent sets are hence characterized by the following fact:

**Lemma**  Inconsistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(ii) for any formula $A \in \mathcal{F}$, $\Delta \vdash A$
Finite Consequence Lemma

We remind here property of the finiteness of the consequence operation.

Lemma  Finite Consequence
For every set $\Delta$ of formulas and for every formula $A \in \mathcal{F}$
$\Delta \vdash A$ if and only if there is a finite set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$

Proof
If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$, hence by the monotonicity of the consequence, also $\Delta \vdash A$
Finite Consequence Lemma

Assume now that $\Delta \vdash A$ and let $A_1, A_2, ..., A_n$ be a formal proof of $A$ from $\Delta$

Let

$$\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$$

Obviously, $\Delta_0$ is finite and $A_1, A_2, ..., A_n$ is a formal proof of $A$ from $\Delta_0$
Finite Inconsistency Theorem

The following theorem is a simple corollary of just proved Finite Consequence Lemma

**Theorem**  Finite Inconsistency

(1.) If a set $\Delta$ is **inconsistent**, then it has a finite inconsistent subset $\Delta_0$

(2.) If every finite subset of a set $\Delta$ is **consistent** then the set $\Delta$ is also **consistent**
Finite Inconsistency Theorem

Proof

If $\Delta$ is **inconsistent**, then for some formula $A$,

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A$$

By the Finite Consequence Lemma, there are finite subsets $\Delta_1$ and $\Delta_2$ of $\Delta$ such that

$$\Delta_1 \vdash A \quad \text{and} \quad \Delta_2 \vdash \neg A$$

The union $\Delta_1 \cup \Delta_2$ is a finite subset of $\Delta$ and by monotonicity

$$\Delta_1 \cup \Delta_2 \vdash A \quad \text{and} \quad \Delta_1 \cup \Delta_2 \vdash \neg A$$

Hence we proved that $\Delta_1 \cup \Delta_2$ is a **finite inconsistent subset** of $\Delta$.

The second implication (2) is the opposite to the one just proved and hence also holds.
The following Lemma links the notion of non-provability and consistency.

It will be used as an important step in our Proof 2 of the Completeness Theorem.

Lemma

For any formula $A \in \mathcal{F}$, if \( \not \vdash A \) then the set \{\neg A\} is consistent.
Consistency Lemma

Proof  We prove the opposite implication
If \( \{\neg A\} \) is inconsistent, then \( \vdash A \)
Assume that \( \{\neg A\} \) is inconsistent
By the Inconsistency Condition Lemma we have that
\( \{\neg A\} \vdash B \) for any formula \( B \), and hence in particular
\[ \{\neg A\} \vdash A \]
By Deduction Theorem we get
\[ \vdash (\neg A \Rightarrow A) \]
We assumed (9.) about the system \( S \) that
\[ ((\neg A \Rightarrow A) \Rightarrow A) \]
By Modus Ponens we get
\[ \vdash A \]
This ends the proof
Complete and Incomplete Sets

Another important notion, is that of a complete set of formulas.

Complete sets, as defined here are sometimes called maximal, but we use the first name for them. They are defined as follows.

Definition Complete set
A set \( \Delta \) of formulas is called complete if for every formula \( A \in F \)

\[ \Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A \]

Godel used this notion of complete sets in his Incompleteness of Arithmetic Theorem

The complete sets are characterized by the following fact.
Complete and Incomplete Sets

Complete Set Condition Lemma
For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent

(i) The set $\Delta$ is complete
(ii) For every formula $A \in \mathcal{F}$,
    if $\Delta \not\vdash A$ then the set $\Delta \cup \{A\}$ is inconsistent

Proof
We consider two cases

Case 1 We show that (i) implies (ii) and
Case 2 we show that (ii) implies (i)
Complete Set Condition Lemma

Proof of Case 1
Assume (i) and not(ii) i.e.
assume that $\Delta$ is complete and there is a formula $A \in \mathcal{F}$
such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent
We have to show that we get a contradiction
But if $\Delta \not\vdash A$, then from the assumption that $\Delta$ is complete
we get that

$$\Delta \vdash \neg A$$

By the monotonicity of the consequence we have that

$$\Delta \cup \{A\} \vdash \neg A$$
Complete Set Condition Lemma

By assumed provability in $S$ of $4. \vdash (A \Rightarrow A)$

By monotonicity $\Delta \vdash (A \Rightarrow A)$ and by Deduction Theorem

$$\Delta \cup \{A\} \vdash A$$

We hence proved that that there is a formula $A \in \mathcal{F}$ such that

$$\Delta \cup \{A\} \quad \text{and} \quad \Delta \cup \{A\} \vdash \neg A$$

i.e. that the set $\Delta \cup \{A\}$ is inconsistent

Contradiction
Complete Set Condition Lemma

Proof of **Case 2**

Assume (ii), i.e. that for every formula $A \in \mathcal{F}$ if $\Delta \not\vdash A$ then the set $\Delta \cup \{A\}$ is inconsistent.

Let $A$ be any formula.

We want to show (i), i.e. to show that the following condition

$$\textbf{C} : \quad \Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A$$

is satisfied.

**Observe** that if

$$\Delta \vdash \neg A$$

then the condition $\textbf{C}$ is obviously satisfied.
Complete Set Condition Lemma

If, on the other hand,

\[ \Delta \not\subseteq \neg A \]

then we are going to show now that it must be, under the assumption of (ii), that \( \Delta \vdash A \) i.e. that (i) holds

Assume that

\[ \Delta \not\subseteq \neg A \]

then by (ii) the set \( \Delta \cup \{\neg A\} \) is inconsistent
The Inconsistency Condition Lemma says
For every set $\Delta \subseteq F$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is inconsistent,

(i) for any formula $A \in F$, $\Delta \vdash A$

We just proved that the set $\Delta \cup \{\neg A\}$ is inconsistent
So by the above Lemma we get

$\Delta \cup \{\neg A\} \vdash A$
Complete Set Condition Lemma

By the Deduction Theorem \( \Delta \cup \{\neg A\} \vdash A \) implies that

\[ \Delta \vdash (\neg A \Rightarrow A) \]

Observe that

\[ ((\neg A \Rightarrow A) \Rightarrow A) \]

is a provable formula 4. in \( S \)

By monotonicity

\[ \Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A) \]

Detaching, by MP the formula \((\neg A \Rightarrow A)\) we obtain that

\[ \Delta \vdash A \]

This ends the proof that (i) holds.
Incomplete Sets

Definition  Incomplete Set
A set \( \Delta \) of formulas is called **incomplete** if it is not **complete** i.e. when the following condition holds

There exists a formula \( A \in \mathcal{F} \) such that

\[
\Delta \not\vDash A \quad \text{and} \quad \Delta \not\vDash \neg A
\]
Incomplete Set Condition Lemma

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of incomplete sets

Lemma Incomplete Set Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) $\Delta$ is incomplete,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$ and the set $\Delta \cup \{A\}$ is consistent.
Main Lemma: Complete Consistent Extension

Now we are going to prove a Lemma that is essential to the construction of the special set $\Delta^*$ mentioned in the Step 1 of the proof of the Completeness Theorem and hence to the proof of the theorem itself.

Let’s first introduce one more notion.
Complete Consistent Extension

**Definition**  Extension $\Delta^*$ of the set $\Delta$

A set $\Delta^*$ of formulas is called an extension of a set $\Delta$ of formulas if the following condition holds

$$\{A \in \mathcal{F} : \Delta \vdash A \} \subseteq \{A \in \mathcal{F} : \Delta^* \vdash A \}$$

i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$

In this case we say also that $\Delta$ extends to the set of formulas $\Delta^*$
Complete Consistent Extension

The Main Lemma

Every consistent set $\Delta$ of formulas can be extended to a complete consistent set $\Delta^*$ of formulas, i.e.

For every consistent set $\Delta$ there is a set $\Delta^*$ that is complete and consistent and is an extension of $\Delta$, i.e.

$$Cn(\Delta) \subseteq Cn(\Delta^*)$$
Proof of the Main Lemma

Proof
Assume that the lemma does not hold, i.e. that there is a consistent set $\Delta$, such that all its consistent extensions are not complete.

In particular, as $\Delta$ is an consistent extension of itself, we have that $\Delta$ is not complete.

The proof consists of a construction of a particular set $\Delta^*$ and proving that it forms a complete consistent extension of $\Delta$.

This is contrary to the assumption that all its consistent extensions are not complete.
Construction of $\Delta^*$

As we know, the set $\mathcal{F}$ of all formulas is enumerable; they can hence be put in an infinite sequence

$$\mathcal{F} \ A_1, A_2, \ldots, A_n, \ldots$$

such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define, by mathematical induction, an infinite sequence

$$D \ \{\Delta_n\}_{n \in \mathbb{N}}$$

of consistent subsets of formulas together with a sequence

$$B \ \{B_n\}_{n \in \mathbb{N}}$$

of formulas as follows.
Construction of $\Delta^*$

Initial Step
In this step we define the sets

$\Delta_1, \Delta_2$ and the formula $B_1$

and prove that

$\Delta_1$ and $\Delta_2$

are consistent, incomplete extensions of $\Delta$

We take as the first set in $D$ the set $\Delta$, i.e. we define

$\Delta_1 = \Delta$
Construction of $\Delta^*$

By assumption the set $\Delta$, and hence also $\Delta_1$ is **not complete**.

From the **Incomplete Set Condition Lemma** we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \not\models B \quad \text{and} \quad \Delta_1 \cup \{B\} \quad \text{is consistent}$$

Let $B_1$ be the **first formula** with this property in the sequence $\mathcal{F}$ of all formulas.

We define

$$\Delta_2 = \Delta_1 \cup \{B_1\}$$
Construction of $\Delta^*$

Observe that the set $\Delta_2$ is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2$$

By monotonicity $\Delta_2$ is a consistent extension of $\Delta$

Hence, as we assumed that all consistent extensions of $\Delta$ are not complete, we get that $\Delta_2$ cannot be complete, i.e.

$$\Delta_2 \text{ is incomplete}$$
Construction of $\Delta^*$

**Inductive Step**

**Suppose** that we have defined a sequence

$\Delta_1, \Delta_2, \ldots, \Delta_n$

of **incomplete, consistent extensions** of $\Delta$ and a sequence

$B_1, B_2, \ldots, B_{n-1}$

of formulas, for $n \geq 2$
Construction of $\Delta^*$

Since $\Delta_n$ is incomplete, it follows from the Incomplete Set Condition Lemma that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_n \not\vdash B \quad \text{and} \quad \Delta_n \cup \{B\} \quad \text{is consistent}$$
Construction of $\Delta^*$

Let $B_n$ be the first formula with this property in the sequence $F$ of all formulas.

We define

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}$$

By the definition

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set $\Delta_{n+1}$ is a consistent extension of $\Delta$.

Hence by our assumption that all consistent extensions of $\Delta$ are incomplete, we get that

$$\Delta_{n+1}$$

is an incomplete consistent extension of $\Delta$. 
Construction of $\Delta^*$

By the principle of mathematical induction we have defined an infinite sequence

$$\Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots.$$  

such that for all $n \in \mathbb{N}$, $\Delta_n$ is consistent, and each $\Delta_n$ an incomplete consistent extension of $\Delta$

Moreover, we have also defined a sequence

$$B_1, B_2, \ldots, B_n, \ldots$$

of formulas, such that for all $n \in \mathbb{N}$,

$$\Delta_n \nvdash B_n \quad \text{and} \quad \Delta_n \cup \{B_n\} \quad \text{is consistent}$$

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$
Definition of $\Delta^*$

Now we are ready to define $\Delta^*$

Definition of $\Delta^*$

$$\Delta^* = \bigcup_{n \in N} \Delta_n$$

To complete the proof our theorem we have now to prove that $\Delta^*$ is a complete consistent extension of $\Delta$
Obviously directly from the definition \( \Delta \subseteq \Delta^* \) and hence we have the following:

**Fact 1** \( \Delta^* \) is an extension of \( \Delta \)

By Monotonicity of Consequence \( Cn(\Delta) \subseteq Cn(\Delta^*) \), hence extension.

As the next step we prove:

**Fact 2** The set \( \Delta^* \) is consistent.
$\Delta^*$ Consistent

**Proof** that $\Delta^*$ is consistent

Assume that $\Delta^*$ is inconsistent

By the Finite Inconsistency Theorem there is a finite subset $\Delta_0$ of $\Delta^*$ that is inconsistent, i.e.

$$\Delta_0 \subseteq \bigcup_{n \in \mathbb{N}} \Delta_n, \quad \Delta_0 = \{C_1, ..., C_n\}, \quad \Delta_0 \text{ is inconsistent}$$
Proof of $\Delta^*$ Consistent

We have $\Delta_0 = \{C_1, \ldots, C_n\}$

By the definition of $\Delta^*$ for each formula $C_i \in \Delta_0$

$$C_i \in \Delta_{k_i}$$

for certain $\Delta_{k_i}$ in the sequence

$$D \quad \Delta = \Delta_1 \subseteq \Delta_2 \subseteq \ldots, \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq \ldots$$

Hence $\Delta_0 \subseteq \Delta_m$ for $m = \max\{k_1, k_2, \ldots, k_n\}$
Proof of $\Delta^*$ Consistent

But we proved that all sets of the sequence $\mathbb{D}$ are consistent

This contradicts the fact that $\Delta_m$ is consistent as it contains an inconsistent subset $\Delta_0$

This contradiction ends the proof that $\Delta^*$ is consistent
Proof of $\Delta^*$ Complete

**Fact 3** The set $\Delta^*$ is complete

**Proof** Assume that $\Delta^*$ is not complete.

By the Incomplete Set Condition, there is a formula $B \in F$ such that

$\Delta^* \not\vdash B$, and the set $\Delta^* \cup \{B\}$ is consistent

By definition of the sequence $D$ and the sequence $B$ of formulas we have that for every $n \in N$

$\Delta_n \not\vdash B_n$ and the set $\Delta_n \cup \{B_n\}$ is consistent

Moreover $B_n \in \Delta_{n+1}$ for all $n \geq 1$
Proof of $\Delta^*$ Complete

Since the formula $B$ is one of the formulas of the sequence $B$ so we get that $B = B_j$ for certain $j$

By definition, $B_j \in \Delta_{j+1}$ and it proves that

$$B \in \Delta^* = \bigcup_{n \in \mathbb{N}} \Delta_n$$

But this means that $\Delta^* \vdash B$

This is a contradiction with the assumption $\Delta^* \nvdash B$ and it ends the proof of the Fact 3

Facts 1-3 prove that that $\Delta^*$ is a complete consistent extension of $\Delta$ and completes the proof out Main Lemma
Proof 2 of Completeness Theorem
Proof 2 of Completeness Theorem

As by assumption our proof system $S$ is sound, we have to prove only the Completeness part of the Completeness Theorem, i.e., to prove that

Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\text{If } \models A, \text{ then } \vdash A$$

We prove it by proving the opposite implication

Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\text{If } \not\models A, \text{ then } \not\vdash A$$
Proof 2 of Completeness Theorem

Proof
Assume that $A$ doesn’t have a proof in $S$, we want to define a counter-model for $A$

But if $\not\vdash A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is consistent

By the Main Lemma there is a complete, consistent extension of the set $\{\neg A\}$

This means that there is a set $\Delta^*$ such that $\{\neg A\} \subseteq \Delta^*$, i.e.

$$E \quad \neg A \in \Delta^* \quad \text{and} \quad \Delta^* \quad \text{is complete and consistent}$$
Proof 2 of Completeness Theorem

Since $\Delta^*$ is a **consistent, complete** set, it satisfies the following form of

Consistency Condition
For any $A \in \mathcal{F}$,

$$\Delta^* \not\vdash A \quad \text{or} \quad \Delta^* \not\vdash \neg A$$

$\Delta^*$ is also **complete** i.e. satisfies

Completeness Condition
For any $A \in \mathcal{F}$,

$$\Delta^* \vdash A \quad \text{or} \quad \Delta^* \vdash \neg A$$
Proof 2 of Completeness Theorem

Directly from the Completeness and Consistency Conditions we get the following

Separation Condition

For any $A \in \mathcal{F}$, exactly one of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or (2) $\Delta^* \vdash \neg A$

In particular case we have that for every propositional variable $a \in \text{VAR}$ exactly one of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or (2) $\Delta^* \vdash \neg a$

This justifies the correctness of the following definition
Proof 2 of Completeness Theorem

Definition
We define the variable truth assignment

\[ v : \text{VAR} \rightarrow \{ T, F \} \]

as follows:

\[ v(a) = \begin{cases} 
T & \text{if } \Delta^* \vdash a \\
F & \text{if } \Delta^* \vdash \neg a.
\end{cases} \]

We show, as a separate Lemma below, that such defined variable assignment \( v \) has the following property
Property of $v$ Lemma

**Lemma**  Property of $v$

Let $v$ be the variable assignment defined above and $v^*$ its extension to the set $\mathcal{F}$ of all formulas $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B \end{cases}$$
Proof 2 of Completeness Theorem

Given the Property of \( \nu \) Lemma (still to be proved),
we now \textbf{prove} that the \( \nu \) is in fact, a \textbf{counter model} for any
formula \( A \), such that \( \not\models A \).

Let \( A \) be such that \( \not\models A \).

By the Property \( E \) we have that \( \neg A \in \Delta^* \).

So obviously
\[ \Delta^* \vdash \neg A \]

Hence by the Property of \( \nu \) Lemma
\[ \nu^*(A) = F \]

what \textbf{proves} that \( \nu \) is a \textbf{counter-model} for \( A \) and it
\textbf{ends the proof} of the Completeness Theorem.
Proof of Property of $\nu$ Lemma

Proof of the Property of $\nu$ Lemma
The proof is conducted by the induction on the degree of the formula $A$

Initial step $A$ is a propositional variable so the Lemma holds by definition of $\nu$

Inductive Step
If $A$ is not a propositional variable, then $A$ is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas $C, D$
By the inductive assumption the Lemma holds for the formulas $C$ and $D$
Case  \( A = \neg C \)

By the **Separation Condition** for \( \Delta^* \) we consider two possibilities

1. \( \Delta^* \vdash A \)
2. \( \Delta^* \vdash \neg A \)

Consider case 1. i.e. we assume that \( \Delta^* \vdash A \)

It means that

\[
\Delta^* \vdash \neg C
\]

Then from the fact that \( \Delta^* \) is **consistent** it must be that

\[
\Delta^* \nvdash C
\]
Proof of Property of $v$ Lemma

By the inductive assumption we have that $v^*(C) = F$ and accordingly $v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T$

Consider case 2. i.e. we assume that $\Delta^* \vdash \neg A$

Then from the fact that $\Delta^*$ is consistent it must be that $\Delta^* \not\vdash A$ and

$\Delta^* \not\vdash \neg C$

If so, then $\Delta^* \vdash C$, as the set $\Delta^*$ is complete

By the inductive assumption, $v^*(C) = T$, and accordingly

$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F$

Thus $A$ satisfies the Property of $v$ Lemma.
Proof of Property of \( v \) Lemma

**Case** \( A = (C \Rightarrow D) \)

As in the previous case, we assume that the Lemma holds for the formulas \( C, D \) and we consider by the *Separation Condition* for \( \Delta^* \) two possibilities:

1. \( \Delta^* \vdash A \) and
2. \( \Delta^* \vdash \neg A \)

**Case 1.** Assume \( \Delta^* \vdash A \)

It means that \( \Delta^* \vdash (C \Rightarrow D) \)

If at the same time \( \Delta^* \not\vdash C \), then \( v^*(C) = F \), and accordingly

\[
\begin{align*}
v^*(A) &= v^*(C \Rightarrow D) = \\
v^*(C) \Rightarrow v^*(D) &= F \Rightarrow v^*(D) = T
\end{align*}
\]
Proof of Property of \( v \) Lemma

If at the same time \( \Delta^* \vdash C \), then since \( \Delta^* \vdash (C \Rightarrow D) \), we infer, by Modus Ponens, that

\[ \Delta^* \vdash D \]

If so, then \( v^*(C) = v^*(D) = T \)

and accordingly

\[ v^*(A) = v^*(C \Rightarrow D) = \]

\[ v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = T \]

Thus if \( \Delta^* \vdash A \), then \( v^*(A) = T \)
Proof of Property of $\nu$ Lemma

**Case 2.** Assume now, as before, that $\Delta^* \vdash \neg A$,

Then from the fact that $\Delta^*$ is **consistent** it must be that $\Delta^* \not\vdash A$, i.e.,

$$\Delta^* \not\vdash (C \Rightarrow D)$$

It follows from this that $\Delta^* \not\vdash D$

For if $\Delta^* \vdash D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula 1. in S, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D))$$

Applying Modus Ponens we obtain

$$\Delta^* \vdash (C \Rightarrow D)$$

which is contrary to the assumption, so it must be $\Delta^* \not\vdash D$
Proof of Property of $v$ Lemma

Also we must have

\[ \Delta^* \vdash C \]

for otherwise, as $\Delta^*$ is complete we would have

\[ \Delta^* \vdash \neg C \]

But this is impossible since the formula \((\neg C \Rightarrow (C \Rightarrow D))\) is assumed to be provable formula 9. in $S$ and by monotonicity

\[ \Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D)) \]

Applying Modus Ponens we would get

\[ \Delta^* \vdash (C \Rightarrow D) \]

which is contrary to the assumption $\Delta^* \not\vdash (C \Rightarrow D)$

This ends the proof of the Property of $v$ Lemma and hence the proof of the Completeness Theorem is also completed.