cse371/mat371
LOGIC

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Short Review for Q4

Q4 Covers Chapter 7

PART 1: DEFINITIONS

PART 2: Problems
PART 1: Definitions from Lecture 7 you have to know for Q4
Definition: Proof System

Definition 1
By a proof system we understand a quadruple

\[ S = (\mathcal{L}, \mathcal{E}, \mathcal{LA}, \mathcal{R}) \]

where

\[ \mathcal{L} = \{ \mathcal{A}, \mathcal{F} \} \] is a language of S with a set \( \mathcal{F} \) of formulas

\[ \mathcal{E} \] is a set of expressions of S

In particular case \( \mathcal{E} = \mathcal{F} \)

\[ \mathcal{LA} \subseteq \mathcal{E} \] is a non-empty, finite set of logical axioms of S

\[ \mathcal{R} \] is a non-empty, finite set of rules of inference of S
Definition: Sound Rule of Inference

Definition 2
An inference rule

\[(r) \quad \frac{P_1; P_2; \ldots; P_m}{C}\]

is sound under a semantics \(M\) if and only if all \(M\)-models of the set \(\{P_1, P_2, \ldots, P_m\}\) of its **premisses** are also \(M\)-models of its **conclusion** \(C\).

In particular, in case of **extensional propositional semantics** when the condition below holds for any truth assignment \(v : VAR \rightarrow LV\):

If \(v \models_M \{P_1, P_2, \ldots, P_m\}\), then \(v \models_M C\)
Definition: Direct Consequence

Definition 3
For any rule of inference $r \in R$ of the form

$$(r) \quad \frac{P_1 ; P_2 ; \ldots ; P_m}{C}$$

$C$ is called a **direct consequence** of $P_1, \ldots P_m$ by virtue of the rule $r \in R$. 
Definition:  Formal Proof

Definition 4

A formal proof of an expression $E \in \mathcal{E}$ in a proof system $S = (\mathcal{L}, \mathcal{E}, LA, R)$ is a sequence

$$A_1, A_2, \ldots, A_n \quad \text{for} \quad n \geq 1$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or $A_i$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference.

$n \geq 1$ is the length of the proof $A_1, A_2, \ldots, A_n$.
NOTATION: Provable Expressions

Notation

We write \( \vdash_S E \) to denote that \( E \in \mathcal{E} \) has a formal proof in the proof system \( S \).

A set

\[
P_S = \{ E \in \mathcal{E} : \vdash_S E \}
\]

is called the set of all provable expressions in \( S \).
Definition: Sound S

Definition 5
Given a proof system

\[ S = (\mathcal{L}, \mathcal{E}, \text{LA}, \mathcal{R}) \]

We say that the system \( S \) is \textbf{sound} under a semantics \( M \) iff the following conditions hold

1. Logical axioms \( \text{LA} \) are \textbf{tautologies} of under the semantics \( M \), i.e.

\[ \text{LA} \subseteq T_M \]

2. Each rule of inference \( r \in \mathcal{R} \) is \textbf{sound} under the semantics \( M \)
THEOREMS: Soundness Theorem

Let \( P_S \) be the set of all provable expressions of \( S \) i.e.

\[
P_S = \{ A \in \mathcal{E} : \vdash_S A \}
\]

Let \( T_M \) be a set of all expressions of \( S \) that are tautologies under a semantics \( M \), i.e.

\[
T_M = \{ A \in \mathcal{E} : \models_M A \}
\]

Our GOAL is to prove the following theorems:

**Soundness Theorem** (for \( S \) and semantics \( M \))

\[
P_S \subseteq T_M
\]

i.e. for any \( A \in \mathcal{E} \), the following implication holds

If \( \vdash_S A \) then \( \models_M A \)
THEOREMS: Completeness Theorem

Completeness Theorem (for $S$ and semantics $M$)

$$P_S = T_M$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \iff \models_M A$$

The Completeness Theorem consists of two parts:
Part 1: Soundness Theorem

$$P_S \subseteq T_M$$

Part 2: Completeness Part of the Completeness Theorem

$$T_M \subseteq P_S$$
PART 2: Simple Problems
Formal Proofs

Problem 1
Given a proof system:

\[ S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \mathcal{R} = \{(r)\} \]

where 
\[
(r) \quad \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}
\]

Write a formal proof in \( S \) with 2 applications of the rule \( (r) \)

Solution: There are many solutions. Here is one of them.

Required formal proof is a sequence \( A_1, A_2, A_3 \), where
\[
A_1 = (A \Rightarrow A) \quad \text{(Axiom)}
\]
\[
A_2 = (A \Rightarrow (A \Rightarrow A))
\]
Rule \( (r) \) application 1 for \( A = A, B = A \)
\[
A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))
\]
Rule \( (r) \) application 2 for \( A = A, B = (A \Rightarrow A) \)
Soudness

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

Problem 2

Prove that \( S \) is sound under classical semantics.

Solution

1. Both axioms of \( S \) are basic classical tautologies
2. Consider the rule of inference of \( S \)

\[ (r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \]

Assume that its premise (the only premise) is true, i.e. let \( v \) be any truth assignment, such that \( v^*(A \Rightarrow B) = T \)

We evaluate logical value of the conclusion under the truth assignment \( v \) as follows

\[ v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T \]

for any \( B \) and any value of \( v^*(B) \)
Given a proof system:

\[ S = (\mathcal{L}_{\neg, \to}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

**Problem 3.**
Write a **formal proof** of your choice in \( S \) with 2 applications of the rule \((r)\)

**Solution**

There many of such proofs, of different length, with different choice if axioms - here is my choice: \( A_1, A_2, A_3 \), where

\( A_1 = (A \Rightarrow A) \)
(Axiom)

\( A_2 = (A \Rightarrow (A \Rightarrow A)) \)

Rule \((r)\) application 1 for \( A = A, B = A \)

\( A_3 = (((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))) \)

Rule \((r)\) application 2 for \( A = A, B = (A \Rightarrow A) \)
Formal Proof

Given a proof system:

\[ S = (\mathcal{L}_{\neg,\Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}), (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \]

Problem 4

1. Prove, by constructing a formal proof that

\[ \vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))) \]

Solution  Required formal proof is a sequence \( A_1, A_2 \), where

\[ A_1 = (A \Rightarrow (\neg A \Rightarrow B)) \]

Axiom

\[ A_2 = (((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))) \]

Rule \((r)\) application for \( A = A, B = (\neg A \Rightarrow B) \)
Soundness Theorem

2. Does above point 1. prove that

\[ \models ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))? \]

Solution

Yes, it does because the system \( S \) is sound and we proved by Mathematical Induction over the length of a proof that if \( S \) is sound, then the Soundness Theorem holds for \( S \)
Soundness

Problem 5
Given a proof system:

\[ S = (L_{\neg, \rightarrow}, F, \{(A \rightarrow A), (A \rightarrow (\neg A \rightarrow B))\}, (r) \frac{(A \rightarrow B)}{(B \rightarrow (A \rightarrow B))}) \]

Prove that \( S \) is not sound under \( K \) semantics

Solution
Axiom \((A \rightarrow A)\) is not a \( K \) semantics tautology
Any truth assignment \( v \) such that \( v^*(A) = \bot \) is a counter-model for it
This proves that \( S \) is not sound under \( K \) semantics