Chapter 7
GENERAL PROOF SYSTEMS

PART 1: Introduction- Intuitive definitions
PART 2: Formal Definition of a Proof System
PART 3: Formal Proofs and Simple Examples
PART 4: Consequence, Soundness and Completeness
PART 5: Decidable and Syntactically Decidable Proof Systems
PART 1: General Introduction
Proof Systems - Intuitive Definition

**Proof systems** are built to prove, it means to **construct** formal proofs of statements formulated in a given language.

The first component of any **proof system** is hence its **formal language** $\mathcal{L}$.

**Proof systems** are inference machines with statements called **provable statements** being their **final products**.
The **starting points** of the **inference machine** of a proof system \( S \) are called its **axioms**.

We distinguish two kinds of axioms: **logical axioms** \( LA \) and **specific axioms** \( SA \).

**Semantical link:** we usually build a **proof systems** for a given **language** and its **semantics** i.e. for a logic defined semantically.
Semantical Link

We always choose as a set of **logical axioms** \( \text{LA} \) some subset of **tautologies**, under a given **semantics**.

We will **consider here** only proof systems with **finite sets** of logical or **specific axioms**, i.e. we will examine only **finitely axiomatizable** proof systems.
We can, and we often do, consider proof systems with languages **without yet established semantics**.

In this case the **logical axioms** $\text{LA}$ serve as description of **tautologies** under a **future semantics** yet to be built.

**Logical axioms** $\text{LA}$ of a proof system $S$ are hence not only **tautologies** under an established **semantics**, but they can also guide us how to define a semantics when it is yet unknown.
Specific Axioms

The **specific axioms** *SA* consist of statements that describe a specific knowledge of an universe we want to use the proof system *S* to prove facts about.

**Specific axioms** *SA* are not universally true.

**Specific axioms** *SA* are true only in the universe we are interested to **describe** and **investigate** by the use of the proof system *S*.
Formal Theory

Given a proof system \( S \) with logical axioms \( LA \)

Specific axioms \( SA \) of the proof system \( S \) is any finite set of formulas that are not tautologies, and hence they are always disjoint with the set of logical axioms \( LA \) of \( S \)

The proof system \( S \) with added set of specific axioms \( SA \) is called a formal theory based on \( S \)
The inference machine of a proof system $S$ is defined by a finite set of inference rules.

The inference rules describe the way we are allowed to transform the information within the system with axioms as a starting point.

We depict it informally on the next slide.
Inference Machine

AXIOMS

↓ ↓ ↓

RULES applied to AXIOMS

↓ ↓ ↓

RULES applied to any expressions above

↓ ↓ ↓

Provable formulas
Semantical Link

Semantical link:

Rules of inference of a system $S$ have to preserve the truthfulness of what they are being used to prove.

The notion of truthfulness is always defined by a given semantics $M$.

Rules of inference that preserve the truthfulness are called sound rules under a given semantics $M$.

Rules of inference can be sound under one semantics and not sound under another.
Soundness Theorem

Goal 1
When developing a proof system $S$ the first goal is prove the following theorem about it and its semantics $M$

**Soundness Theorem**
For any formula $A$ of the language of the system $S$
If a formula $A$ is *provable* from *logical axioms* $LA$ of $S$ only, then $A$ is a *tautology* under the semantics $M$
Propositional Proof Systems

We discuss here first only proof systems for propositional languages and call them proof systems for different propositional logics.

Remember

The notion of soundness is connected with a given semantics.

A proof system $S$ can be sound under one semantics, and not sound under the other.

For example a set of axioms and rules sound under classical logic semantics might not be sound under $Ł$ logic semantics, or $K$ logic semantics, or others.
Completeness of the Proof Systems

In general there are many proof systems that are sound under a given semantics, i.e. there are many sound proof systems for a given logic semantically defined.

Given a proof system $S$ with logical axioms $LA$ that is sound under a semantics $M$.

Notation

Denote by $T_M$ the set of all tautologies defined by the semantics $M$, i.e. we have that

$$T_M = \{A \in \mathcal{F} : \models_M A\}$$
A natural question arises:
Are all tautologies i.e formulas $A \in T_M$ provable in the system $S$ ??

We assume that we have already proved that $S$ is sound under the semantics $M$

The positive answer to this question is called completeness property of the system $S$. 

Completeness Property
Completeness Theorem

Goal 2
Given for a sound proof system $S$ under its semantics $M$, our the second goal is to prove the following theorem about $S$

Completeness Theorem
For any formula $A$ of the language of $S$
$A$ is provable in $S$ iff $A$ is a tautology under the semantics $M$
We write the Completeness Theorem symbolically as

$$\vdash_S A \iff \models_M A$$

Completeness Theorem is composed of two parts:
Soundness Theorem and the Completeness Part that proves the completeness property of a sound proof system
Proving Soundness and Completeness

Proving the Soundness Theorem for $S$ under a semantics $M$ is usually a straightforward and not a very difficult task.

We first prove that all logical axioms $LA$ are tautologies, and then we prove that all inference rules of the system $S$ preserve the notion of the truth.

Proving the completeness part of the Completeness Theorem is always a crucial, difficult and sometimes impossible task.
OUR PLAN

We will study two proofs of the **Completeness Theorem** for classical propositional proof system in Chapter 9.

We will present a constructive proofs of **Completeness Theorem** for two different Gentzen style automated theorem proving systems for classical Logic in Chapter 11.

We discuss the **Intuitionistic Logic** in Chapter 12.

**Predicate Logics** are discussed Chapters 13, 14. Kripke Models for **Modal Logics** and **Intuitionistic Logic** are presented in Chapter 15.
PART 2

PROOF SYSTEMS: Formal Definitions
Proof System S

In this section we present formal definitions of the following notions:

Proof system \( S \)

Formal proof from logical axioms in a proof system \( S \)

Formal proof from specific axioms in a proof system \( S \)

Formal Theory based on a proof system \( S \)

We also give examples of different simple proof systems.
Components: Language

Language $\mathcal{L}$ of a proof system $S$ is any formal language $\mathcal{L}$

$$\mathcal{L} = (\mathcal{A}, \mathcal{F})$$

We assume as before that both sets $\mathcal{A}$ and $\mathcal{F}$ are enumerable, i.e. we deal here with enumerable languages.

The Language $\mathcal{L}$ can be propositional or first order (predicate) but we discuss propositional languages first.
Components: Expressions

Expressions $\mathcal{E}$ of a proof system $\mathcal{S}$

Given a set $\mathcal{F}$ of well formed formulas of the language $\mathcal{L}$ of the system $\mathcal{S}$

We often extend the set $\mathcal{F}$ to some set $\mathcal{E}$ of expressions build out of the language $\mathcal{L}$ and some extra symbols, if needed.

In this case all other components of $\mathcal{S}$ are also defined on basis of elements of the set of expressions $\mathcal{E}$.

In particular, and most common case we have that $\mathcal{E} = \mathcal{F}$.
Automated theorem proving systems usually use as their basic components different sets of expressions build out of formulas of the language $L$.

In Chapters 11 and 13 we consider finite sequences of formulas instead of formulas, as basic expressions of the proof systems $RS$ and $RQ$.

We also present there proof systems that use yet other kind of expressions, called original Gentzen sequents or their modifications.

Some systems use yet other expressions such as clauses, sets of clauses, or sets of formulas, others use yet still different expressions.
Semantical Link

We always have to **extend** a given semantics $M$ for the language $L$ of the system $S$ to the set $E$ of all **expression** of the system $S$

Sometimes, like in case of **Resolution** based proof systems we have also to **prove** a **semantic equivalency** of new created expressions $E$ (sets of clauses in Resolution case) with appropriate formulas of $L$
For example, in the automated theorem proving system RS presented in Chapter 11 the basic expressions \( E \) are finite sequences of formulas of \( L = L\{\neg, \cap, \cup, \Rightarrow\} \).

We extend our classical semantics for \( L \) to the set \( F^* \) of all finite sequences of formulas as follows:

For any \( v : \text{VAR} \rightarrow \{F, T\} \) and any \( \Delta \in F^* \), \( \Delta = A_1, A_2, .. A_n \), we put

\[
v^*(\Delta) = v^*(A_1, A_2, .. A_n) = v^*(A_1) \cup v^*(A_2) \cup ... \cup v^*(A_n)
\]

i.e. in a shorthand notation

\[
\Delta \equiv (A_1 \cup A_2 \cup ... \cup A_n)
\]
Components: Logical Axioms

Logical axioms \( LA \) of \( S \) form a non-empty subset of the set \( E \) of expressions of the proof system \( S \), i.e.

\[
LA \subseteq E
\]

In particular, \( LA \) is a non-empty subset of formulas, i.e.

\[
LA \subseteq F
\]

We assume here that the set \( LA \) of logical axioms is always finite, i.e. that we consider here finitely axiomatizable systems.

In general, we assume that the set \( LA \) is primitively recursive i.e. that there is an effective procedure to determine whether a given expression \( E \in E \) is or is not in \( AL \).
Semantical link
Given a semantics $\mathbf{M}$ for $\mathbf{L}$ and its extension to the set $\mathbf{E}$ of all expressions
We extend the notion of tautology to the expressions and write

$$\models_{\mathbf{M}} E$$

to denote that the expression $E \in \mathbf{E}$ is a tautology under semantics $\mathbf{M}$ and we put

$$\mathbf{T}_{\mathbf{M}} = \{ E \in \mathbf{E} : \models_{\mathbf{M}} E \}$$

Logical axioms $\mathbf{LA}$ are always a subset of expressions that are tautologies of under the semantics $\mathbf{M}$, i.e.

$$\mathbf{LA} \subseteq \mathbf{T}_{\mathbf{M}}$$
Components: Rules of Inference

Rules of inference $\mathcal{R}$
We assume that a proof system contains only a finite number of inference rules

We assume that each rule has a finite number of premises and one conclusion

We also assume that one can effectively decide, for any inference rule, whether a given string of expressions form its premises and conclusion or do not, i.e. that All rules $r \in \mathcal{R}$ are primitively recursive
Components: Rules of Inference

Definition

Each rule of inference $r \in \mathcal{R}$ is a relation defined in the set $\mathcal{E}^m$, where $m \geq 1$ with values in $\mathcal{E}$, i.e.

$$r \subseteq \mathcal{E}^m \times \mathcal{E}$$

Elements $P_1, P_2, \ldots, P_m$ of a tuple $(P_1, P_2, \ldots, P_m, C) \in r$ are called premisses of the rule $r$ and $C$ is called its conclusion.

All $r \in \mathcal{R}$ are primitively recursive relations.
Components: Rules of Inference

We write the **inference rules** in a following convenient way

**One** premiss rule

\[
\begin{align*}
(r) & \quad \frac{P_1}{C}
\end{align*}
\]

**Two** premisses rule

\[
\begin{align*}
(r) & \quad \frac{P_1 \quad ; \quad P_2}{C}
\end{align*}
\]

**m** premisses rule

\[
\begin{align*}
(r) & \quad \frac{P_1 \quad ; \quad P_2 \quad ; \quad \ldots \quad ; \quad P_m}{C}
\end{align*}
\]
Semantic Link: Sound Rules of Inference

Given some \( m \) premisses rule

\[
(r) \quad \frac{P_1 ; P_2 ; \ldots ; P_m}{C}
\]

Semantical link

Given a semantics \( M \) for the language \( L \) and for the set of expressions \( E \)

We want the rules of inference \( r \in R \) to preserve truthfulness i.e. to be sound under the semantics \( M \)
General Definition: Sound Rule of Inference

Definition
Given an inference rule \( r \in \mathcal{R} \)

\[
(r) \quad \frac{P_1 ; P_2 ; \ldots ; P_m}{C}
\]

We say that the inference rule \( r \in \mathcal{R} \) is **sound** under a semantics \( \mathcal{M} \) if and only if all \( \mathcal{M} \)-models of the set \( \{P_1, P_2, \ldots, P_m\} \) of its **premisses** are also \( \mathcal{M} \)-models of its **conclusion** \( C \)
Propositional Definition: Sound Rule of Inference

In propositional languages case, the semantics $M$, and hence the $M$-models are defined in terms of the truth assignment $v: VAR \rightarrow LV$, where $LV$ is the set of logical values for the semantics $M$

**Definition**

An inference rule $r \in R$, such that

\[ (r) \quad \frac{P_1 ; P_2 ; \ldots ; P_m}{C} \]

is sound under a semantics $M$ if and only if

the condition below holds or any $v: VAR \rightarrow LV$

If $v \models_M \{P_1, P_2, \ldots, P_m\}$, then $v \models_M C$
Example

Given a rule of inference

\[(r) \quad \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}\]

Prove that \((r)\) is sound under classical semantics

Let \(v\) be any truth assignment, such that \(v \models (A \Rightarrow B)\), i.e. by definition \(v^*(A \Rightarrow B) = T\)

We evaluate logical value of the conclusion under \(v\) as follows

\[v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T\]

for any \(B\) and any value of \(v^*(B)\)

This proves that \(v \models (B \Rightarrow (A \Rightarrow B))\) and hence the soundness of \((r)\)
Formal Definition: Proof System

Definition

By a **proof system** we understand a quadruple

\[ S = (L, E, LA, R) \]

where

- \( L = \{ A, F \} \) is a **language** of S with a set \( F \) of formulas
- \( E \) is a set of **expressions** of S
- In particular case \( E = F \)
- \( LA \subseteq E \) is a **non-empty, finite set** of logical axioms of S
- \( R \) is a **non-empty, finite set** of rules of inference of S
PART 3: Formal Proofs
Simple Examples of Proof Systems
Provable Expressions

A final product of a single or multiple use of the inference rules of $S$, with axioms taken as a starting point are called provable expressions of the proof system $S$.

A single use of an inference rule is called a direct consequence.

A multiple application of rules of inference with axioms taken as a starting point is called a proof.
Definition: Direct Consequence

Formal definitions are as follows

Direct consequence
For any rule of inference $r \in R$ of the form

$$
\begin{array}{c}
P_1 ; P_2 ; \ldots ; P_m \\
\hline
C
\end{array}
$$

$C$ is called a **direct consequence** of $P_1, \ldots P_m$ by virtue of the rule $r \in R$
Definition: Formal Proof

Formal Proof of an expression $E \in \mathcal{E}$ in a proof system

$S = (\mathcal{L}, \mathcal{E}, \mathcal{LA}, \mathcal{R})$

is a sequence $A_1, A_2, \ldots, A_n$ for $n \geq 1$

of expressions from $\mathcal{E}$, such that

$A_1 \in \mathcal{LA}, \quad A_n = E$

and for each $1 < i \leq n$, either $A_i \in \mathcal{LA}$ or $A_i$ is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference

$n \geq 1$ is the length of the proof $A_1, A_2, \ldots, A_n$
Formal Proof Notation

We write

\[ \vdash_S E \]

to denote that \( E \in \mathcal{E} \) has a proof in \( S \)

When the proof system \( S \) is fixed we write \( \vdash E \)

Any \( E \in \mathcal{E} \), such that \( \vdash_S E \) is called a provable expression of \( S \)

The set of all provable expressions of \( S \) is denoted by \( P_S \), i.e. we put

\[ P_S = \{ E \in \mathcal{E} : \vdash_S E \} \]
Simple System $S_1$

Example 1
Consider a very simple proof system system $S_1$ with $\mathcal{E} = \mathcal{F}$

$$S_1 = (\mathcal{L}_{\{P, \Rightarrow\}}, \mathcal{F}, LA = \{(A \Rightarrow A)\}, \mathcal{R} = \{(r) \frac{B}{PB}\})$$

where $A, B \in \mathcal{F}$ are any formulas and

where $P$ is some one argument connective;

we might read $PA$ for example as ”it is possible that $A$”

Observe that even the system $S_1$ has only one axiom, it represents an infinite number of formulas.

We call such axiom an **axiom schema**
Simple System $S_2$

**Example 2**
Consider now a system $S_2$

$$S_2 = (\mathcal{L}_{\{P, \Rightarrow\}}, \mathcal{F}, \{(a \Rightarrow a)\}, (r) \frac{B}{PB})$$

where $a \in \text{VAR}$ is any variable (atomic formula) and $B \in \mathcal{F}$ is any formula.

Observe that the system $S_2$ also has only one axiom similar to the axiom of $S_1$ and they have the same rule of inference but they are different proof systems as for example a formula

$$(((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

is an axiom of system $S_1$ but is not an axiom of the system $S_2$. 

Some Provable Formulas

Example 3
We have that

$$\vdash_{S_1} (((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))))$$

because

$$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))) \in LA \text{ of } S_1$$

other provable formulas are

$$\vdash_{S_1} P(a \Rightarrow a), \quad \vdash_{S_2} P(a \Rightarrow a),$$

$$\vdash_{S_1} PP(a \Rightarrow a), \quad \vdash_{S_2} PP(a \Rightarrow a)$$
Formal Proofs

Formal proofs in both systems of above formulas are identical and are as follows.

**Formal proof** of $P(a \Rightarrow a)$ in $S_1$ and $S_2$ is:

\[
A_1 = (a \Rightarrow a), \quad A_2 = P(a \Rightarrow a)
\]

axiom rule application for $B = (a \Rightarrow a)$
Formal Proofs

Formal proof of $PP(a \Rightarrow a)$ in $S_1$ and $S_2$ is:

$A_1 = (a \Rightarrow a)$, $A_2 = P(a \Rightarrow a)$, $A_3 = PP(a \Rightarrow a)$

- Axiom
- Rule application for $B = (a \Rightarrow a)$
- Rule application for $B = P(a \Rightarrow a)$
Formal Proofs

Exercise
Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \mathcal{R} = \{(r)\} \]

where \( (r) \) \[
\begin{align*}
&\frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))} \\
&\text{Rule (r) application 1 for } A = A, B = A \\
&\text{Axiom}
\end{align*}
\]

Write a formal proof in \( S \) with 2 applications of the rule \( (r) \)

Solution: There are many solutions. Here is one of them.

Required formal proof is a sequence \( A_1, A_2, A_3 \), where
\[
A_1 = (A \Rightarrow A)
\]
(Axiom)
\[
A_2 = (A \Rightarrow (A \Rightarrow A))
\]
Rule \( (r) \) application 1 for \( A = A, B = A \)
\[
A_3 = (((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A))))
\]
Rule \( (r) \) application 2 for \( A = A, B = (A \Rightarrow A) \)
Simple System $S_3$

Consider a very simple proof system system $S_3$ defined as follows

$$S_3 = \left( \mathcal{L}_{\{P, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A)\}, (r_1) \frac{B}{PB}, (r_2) \frac{A ; B}{P(A \Rightarrow B)} \right)$$

Exercise

Write two formal proofs in $S_3$ both of the lengths 4, one of which must contain at least one application of the rule $r_2$
PART 4:  Hypothesis, Consequence, Soundness and Completeness
Proof from Hypothesis

While proving expressions we often use some extra information available, besides the axioms of the proof system. This extra information is called hypothesis in the proof.

Let $\Gamma \subseteq \mathcal{E}$ be a set expressions called hypothesis.

A proof of $E \in \mathcal{E}$ from the set of hypothesis $\Gamma$ in $S$ is a formal proof in $S$, where the expressions from $\Gamma$ are treated as additional hypothesis added to the set $\text{LA}$ of the logical axioms of the system $S$.

Notation: $\Gamma \vdash_S A$

We read it: $A$ has a proof in $S$ from the set $\Gamma$ (and logical axioms LA).
Definition: Proof from Hypothesis

We say that \( A \) has a proof in \( S \) from the set \( \Gamma \) (and logical axioms \( LA \)) if and only if there is a sequence \( A_1, ..., A_n \) of expressions from \( E \), such that
\[
A_1 \in LA \cup \Gamma, \quad A_n = A
\]
and for each \( 1 < i \leq n \), either \( A_i \in LA \cup \Gamma \) or \( A_i \) is a direct consequence of some of the preceding expressions by virtue of one of the rules of inference. We denote it as \( \Gamma \vdash_S A \)
Special Cases

We usually consider and use the case when the set of hypothesis is finite.

**Case** of $\Gamma \subseteq \mathcal{E}$ finite set and $\Gamma = \{B_1, B_2, \ldots, B_n\}$

We use notation

$$B_1, B_2, \ldots, B_n \vdash_S A$$

for $\{B_1, B_2, \ldots, B_n\} \vdash_S A$

**Case** of $\Gamma = \emptyset$ is also a special one.

By the definition of a proof of $A$ from $\Gamma$, $\emptyset \vdash A$ means that in the proof of $A$ we use only axioms $LA$ of $S$

We hence use notation $\vdash_S A$

to denote that $A$ has a proof from empty $\Gamma$; i.e. $A$ has a proof from logical axioms only.
Definition: Consequences of $\Gamma$

For any $\Gamma \subseteq \mathcal{E}$, and $A \in \mathcal{E}$, if $\Gamma \vdash_S A$, then $A$ is called a consequence of $\Gamma$ in $S$.

Definition

We denote by $\text{Cn}_S(\Gamma)$ the set of all consequences of $\Gamma$ in $S$, i.e. we put

$$\text{Cn}_S(\Gamma) = \{ A \in \mathcal{E} : \Gamma \vdash_S A \}$$
Definition: Consequence Operation

Observe that by defining a consequence of $\Gamma$ in $S$, we define in fact a function which to every set $\Gamma \subseteq \mathcal{E}$ assigns a set of all its consequences $Cn_S(\Gamma)$.

We denote this function by $Cn_S$ and adopt the following Definition:

Any function

$$Cn_S : 2^\mathcal{E} \rightarrow 2^\mathcal{E}$$

such that for every $\Gamma \in 2^\mathcal{E}$

$$Cn_S(\Gamma) = \{A \in \mathcal{E} : \Gamma \vdash_S A\}$$

is called the consequence operation in $S$. 
Consequence Operation: Monotonicity

Take any consequence operation

\[ C_{n_S} : 2^E \rightarrow 2^E \]

Monotonicity Property

For any sets \( \Gamma, \Delta \) of expressions of \( S \),

if \( \Gamma \subseteq \Delta \) then \( C_{n_S}(\Gamma) \subseteq C_{n_S}(\Delta) \)

Exercise: write the proof;

it follows directly from the definition of \( C_{n_S} \) and definition of the formal proof
Consequence Operation: Transitivity

Take any consequence operation

\[ \text{Cn}_S : 2^E \rightarrow 2^E \]

Transitivity Property

For any sets \( \Gamma_1, \Gamma_2, \Gamma_3 \) of expressions of \( S \),
if \( \Gamma_1 \subseteq \text{Cn}_S(\Gamma_2) \) and \( \Gamma_2 \subseteq \text{Cn}_S(\Gamma_3) \),
then \( \Gamma_1 \subseteq \text{Cn}_S(\Gamma_3) \)

Exercise: write the proof;
it follows directly from the definition of \( \text{Cn}_S \) and definition of
the formal proof
Consequence Operation: Finiteness

Take any consequence operation

\[ C_n^S : 2^\mathcal{E} \rightarrow 2^\mathcal{E} \]

Finiteness Property

For any expression \( A \in \mathcal{E} \) and any set \( \Gamma \subseteq \mathcal{E} \),

\( A \in C_n^S(\Gamma) \) if and only if there is a finite subset \( \Gamma_0 \) of \( \Gamma \)
such that \( A \in C_n^S(\Gamma_0) \)

Exercise: write the proof;

it follows directly from the definition of \( C_n^S \) and definition of the formal proof
Definition: Sound $S$

Definition

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

We say that the system $S$ is **sound** under a semantics $M$ iff the following conditions hold

1. $LA \subseteq T_M$
2. Each rule of inference $r \in \mathcal{R}$ is **sound**
Example

Given a proof system:

\[ S = (\mathcal{L}_{\neg, \Rightarrow}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}) \]

1. Prove that \( S \) is **sound** under **classical semantics**
2. Prove that \( S \) is **not sound** under **K semantics**
Example

1. Both axioms of $S$ are basic classical tautologies and we have just proved that the rule of inference $(r)$ is sound, hence $S$ is sound.

2. Axiom $(A \Rightarrow A)$ is not a $K$ semantics tautology. Any truth assignment $v$ such that $v^*(A) = \bot$ is a counter-model for it. This proves that $S$ is not sound under $K$ semantics.
Soundness Theorem

Let $P_S$ be the set of all provable expressions of $S$ i.e.

$$P_S = \{ A \in E : \vdash_S A \}$$

Let $T_M$ be a set of all expressions of $S$ that are tautologies under a semantics $M$, i.e.

$$T_M = \{ A \in E : \vdash_M A \}$$

Soundness Theorem for $S$ and semantics $M$

$$P_S \subseteq T_M$$

i.e. for any $A \in E$, the following implication holds

**Exercise:** prove by Mathematical Induction over the length of a proof that if $S$ is sound, the Soundness Theorem holds for $S$
Completeness Theorem

Completeness Theorem for $S$ and semantics $M$

$$\mathcal{P}_S = T_M$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \iff \models_M A$$

The Completeness Theorem consists of two parts:

Part 1: Soundness Theorem

$$\mathcal{P}_S \subseteq T_M$$

Part 2: Completeness Part of the Completeness Theorem

$$T_M \subseteq \mathcal{P}_S$$
Formal Theory

Let a set $SA \subseteq \mathcal{E}$ be such that

$$SA \cap T_M = \emptyset$$

A Formal Theory with the set of specific axioms $SA$ is denoted by $Th_S(SA)$ and defined as follows

$$Th_S(SA) = \{ A \in \mathcal{E} : \ SA \vdash_S A \}$$
Soundness Theorem for formal theory $Th_S(SA)$ based on a sound system $S$ says:

For any formula $A$ of the language of the theory $Th_S(SA)$,

If a formula $A$ is provable in the theory $Th_S(SA)$, then $A$ is true in any model of the set of specific axioms $SA$. 
Completeness of the Theory $Th_S(SA)$

The **Completeness Theorem** for the proof system $S$ established equivalency of the notion of provability and tautology:

$$P_S = T_M$$

Observe holds for theories $Th_S(SA)$ only when $SA = \emptyset$

We nevertheless talk about Complete Theories!
Completeness of the Theory $Th_S(SA)$

Here is the definition of a notion of completeness for proof system $S$ with a language that contains negation $\neg$:

**Definition of a Complete Theory**

A formal theory $Th_S(SA)$ is **complete** iff for any $A$ of the language of the theory the following holds:

$$A \in Th_S(SA) \text{ or } \neg A \in Th_S(SA)$$

The **completeness** of a theory means that we can **prove** or **disapprove** any statement formulated within it.

It hence corresponds to the natural meaning of the word "complete" information.
Consistency of the Theory $Th_S(SA)$

Definition of a Consistent Theory
Let $S$ be a proof system with a language that contains negation $\neg$

A formal theory $Th_S(SA)$ is consistent iff there is no expression $A$ of the language of the theory such that

$$A \in Th_S(SA) \quad \text{and} \quad \neg A \in Th_S(SA)$$

Inconsistency and Completeness are two most important properties of any theory
We use the notion of a formal theory to formalize it and to be able to reason about it
PART 5: Decidable and Syntactically Decidable Proof Systems
Decidable and Syntactically Decidable Proof Systems

A proof system $S$ is called **decidable** when there is a finite, mechanical method for determining, given any expression $A \in \mathcal{E}$ whether there is a proof of $A$ in $S$; i.e. whether $A \in \mathcal{P}_S$

otherwise $S$ is called **undecidable**

**Observe** that the above notion of decidability of the system does not require to find a proof

It requires only a mechanical procedure of deciding whether a proof exists for any expression of the system.
Example

We **prove now** that a Hilbert style proof system $S$ for classical propositional logic presented in Chapter 9 is decidable. We first prove the Completeness Theorem for it:

$$PS = TM$$

We get that for any $A \in \mathcal{E}$

$$A \not\in PS \iff A \not\in TM$$

We have proved already that the notion of classical propositional tautology, i.e. the statement $A \not\in TM$ is decidable.

**We conclude:** the system $S$ is decidable.
Syntactically Decidable Systems

A proof system $S$ is **syntactically decidable** if it is possible to define for it a finite, mechanical method that generates a proof for any given expression $A$ of $S$ otherwise the system $S$ is not syntactically decidable

We call such syntactically decidable systems **automated theorem proving** systems
Syntactically Decidable Systems

All Gentzen type proof systems presented here are both decidable or semi-decidable and syntactically decidable or syntactically semi-decidable.

We usually call them automated theorem proving systems for different logics under consideration.

Resolution based proof systems are also wildly known examples of the syntactically decidable, or semi-decidable systems.

Finding a Gentzen Type, or Resolution type formalization for a given logic is a standard question one asks about any logic being developed.
Formal Proofs

Remember that the notion of a *formal proof* in a system $S$ is *purely syntactical* in its nature. Formal Proof carries a semantical meaning via established semantics and the *Soundness Theorem*. The rules of inference of a proof system define only how to transform strings of symbols of the language into another string of symbols.

The *formal proof*, by the definition says that in order to *prove* an expression $A$ in a system $S$ one has to construct of a *sequence* of proper transformations, *defined* by the rules of inference.
Consider a very simple proof system system $S_1$ with $E = F$

$$S_1 = (L_{\{P, \Rightarrow\}}, \ F, LA1 = \{(A \Rightarrow A)\}, (r) \frac{B}{PB})$$

where $A, B \in F$ are any formulas and
where $P$ is some one argument connective;
we might read $PA$ for example as ”it is possible that $A”
Observe that even the system $S_1$ has only one axiom, it represents an infinite number of formulas.
We call such axiom **axiom schema**
Simple System $S_2$

Consider now a system $S_2$

\[ S_2 = (\mathcal{L}_{\{P, \Rightarrow\}}, \mathcal{F}) \quad \text{LA2} = \{(a \Rightarrow a)\}, (r) \quad \frac{B}{PB}, \]

where $a \in VAR$ is any variable (atomic formula) and $B \in \mathcal{F}$ is any formula.

Observe that even the system $S_1$ has only one axiom, it is also an **axiom schema**

Observe that for example a formula

\[ ((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))) \]

is an axiom of system $S_1$

but is not an axiom of the system $S_2$
Some Provable Formulas

We have that

\[ \vdash_{S_1} \left( (Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)) \right) \]

because

\[ \left( (Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)) \right) \in LA1 \]

other provable formulas are

\[ \vdash_{S_1} P(a \Rightarrow a), \quad \vdash_{S_2} P(a \Rightarrow a), \]

\[ \vdash_{S_1} PP(a \Rightarrow a), \quad \vdash_{S_2} PP(a \Rightarrow a) \]
Formal Proofs

Formal proofs in both systems of above formulas are identical and are as follows.

**Formal proof** of $P(a \Rightarrow a)$ in $S_1$ and $S_2$ is:

$A_1 = (a \Rightarrow a)$, \hspace{1cm} $A_2 = P(a \Rightarrow a)$

axiom \hspace{1cm} rule application

for $B = (a \Rightarrow a)$
Formal Proofs

Formal proof of \( PP(a \Rightarrow a) \) in \( S_1 \) and \( S_2 \) is:

\[
A_1 = (a \Rightarrow a), \quad A_2 = P(a \Rightarrow a), \quad A_3 = PP(a \Rightarrow a)
\]

- \( A_1 = (a \Rightarrow a) \), axiom
- \( A_2 = P(a \Rightarrow a) \), rule application for \( B = (a \Rightarrow a) \)
- \( A_3 = PP(a \Rightarrow a) \), rule application for \( B = P(a \Rightarrow a) \)
Proof Search

Let’s search for a proof (if exists) of the formula $A$ below in $S_2$

$$A = PP(((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))))$$

Observe, that if $A$ had the proof, the only last step in this proof would be the application of the rule $(r) \frac{B}{PB}$ to the formula $P(((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))))$

This formula, in turn, if it had the proof, the only last step in its proof would be the application of the rule $r$ to the formula $(((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))))$

The search process stops here
Proof Search

Observe that

$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))) \not\in LA2$

what means that our search for the proof has failed; i.e. our found sequence of formulas does not constitute a proof.

Moreover, the search was, at each step unique what proves that the proof of $A$ in $S_2$ does not exist, i.e.

$\kappa_{S_2} PP(((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))))$
We easily generalize above example to a proof search procedure to any formula $A$ of $S_1$ or $S_2$ as follows:

**Procedure SP**

**Step**: Check the main connective of $A$

If main connective is $P$, it means that $A$ was obtained by the rule $r$

**Erase** the main connective $P$

**Repeat** until no $P$ as a main connective is left.

If the main connective is $\Rightarrow$, check if a formula is an axiom

If it is an axiom, **stop** and **yes** we have a proof

If it is not an axiom, **stop** and **no**, proof does not exist
Syntactical Decidability

The **Procedure SP** is a *finite, effective, automatic* procedure of searching for a proof of formulas in both our proof systems. This proves the following.

**Fact** Proof systems $S_1$ and $S_2$ are *syntactically decidable*
Semantical link

**Remark** that we haven’t defined a **semantics** for the language $\mathcal{L}_{\{\Rightarrow, P\}}$ of systems $S_1, S_2$

We can’t talk about the **soundness** of these systems yet but we can think how to define a sound semantics for our systems.

If we want to understand statement $PA$ as “A is possible” we need to define some kind of **modal** semantics.
Semantical link

All known modal semantics **extend** the classical semantics, i.e. they are **the same** as classical one on **non-modal** connectives

Hence under any possible modal semantics axioms $S_1, S_2$ of would be a **sound axiom** under standard modal logics semantics, as they are classical tautologies.

To **assure the soundness** of both systems we must have a modal semantics $M$ that makes the rule

$$(r) \quad \frac{B}{PB}$$

**sound** under the modal semantics $M$
General Question 1

**General Q1:** Are all proof systems decidable?  
**Answer Q1:** No, not all proof systems are decidable.  
The most "natural" and historically first developed proof system for classical predicate logic is **not decidable**.
General Question 2

**General Q2**  Can we give an example of a logic and its complete proof system which is not decidable, but the logic does have another complete, syntactically decidable proof system?

**Answer Q2:**  Hilbert style proof system for classical propositional logic complete and decidable but is not syntactically decidable

We present in Chapter 9 two complete proof systems for classical propositional logic that are syntactically decidable