Chapter 3
Propositional Languages

PART 1: Propositional Languages: Intuitive Introduction
PART 2: Propositional Languages: Formal Definitions
We define now a general notion of a propositional language. We show how to obtain, as specific cases, various languages for propositional classical logic and some non-classical logics. We assume the following:

All propositional languages contain an infinitely countable set of variables \( \text{VAR} \), which elements are denoted by

\[ a, b, c, \ldots \]

with indices, if necessary.

All propositional languages share the general way their sets of formulas are formed.
Propositional Languages

We distinguish one propositional language from the other is the choice of its set of propositional connectives.

We adopt a notation

$L_{\text{CON}}$

where $CON$ stands for the set of connectives.

We use a notation

$L$

when the set of connectives is fixed.
Propositional Languages

For example, the language

\[ L\{\neg}\]  

denotes a propositional language with only one connective \( \neg \).

The language

\[ L\{\neg, \Rightarrow}\]  

denotes that a language with two connectives \( \neg \) and \( \Rightarrow \) adopted as propositional connectives.

**Remember:** formal languages deal with symbols only and are also called *symbolic languages*.
General Principles

Symbols for connectives do have intuitive meaning.
Semantics provides a formal meaning of the connectives and is defined separately.
One language can have many semantics.
Different logics can share the same language.
For example: the language

\[ L\{\neg,\cap,\cup,\Rightarrow}\]

is used as a propositional language of classical and intuitionistic logics, some many-valued logics, and we extend it to the language of many modal logics.
General Principles

Several languages can share the same semantics. The classical propositional logic is the best example of such situation. Due to the functional dependency of classical logic connectives the languages:

\[ \mathcal{L}_{\neg, \Rightarrow}, \mathcal{L}_{\neg, \cap}, \mathcal{L}_{\neg, \cup}, \mathcal{L}_{\neg, \cap, \cup, \Rightarrow}, \mathcal{L}_{\neg, \cap, \cup, \Rightarrow, \Leftrightarrow}, \mathcal{L}_{\uparrow}, \mathcal{L}_{\downarrow} \]

are all equivalent under the classical semantics. We will define formally languages equivalency of languages in the next chapter.
General Principles

Propositional connectives have well established names and the way we read them, even if their semantics may differ. We use names negation, conjunction, disjunction and implication for $\neg$, $\cap$, $\cup$, $\Rightarrow$, respectively. The connective $\uparrow$ is called alternative negation and $A \uparrow B$ reads: not both $A$ and $B$. The connective $\downarrow$ is called joint negation and $A \downarrow B$ reads: neither $A$ nor $B$. 
Some Non-Classical Propositional Connectives

Other most common propositional connectives are modal connectives of possibility and necessity.

Modal connectives are not extensional.

Standard modal symbols are: □ for necessity and ♦ for possibility.

We will also use symbols C and I for modal connectives of possibility and necessity, respectively.

The formula CA, or ♦A reads: it is possible that A or A is possible and

The formula IA, or □A reads: it is necessary that A or A is necessary.
Modal Propositional Connectives

Symbols $C$ and $I$ are used for their topological meaning in the semantics of standard modal logics S4 and S5.

In topology $C$ is a symbol for a set closure operation. $CA$ means a closure of a set $A$.

$I$ is a symbol for a set interior operation. $IA$ denotes an interior of the set $A$.

Modal logics extend the classical logic.

A modal logic languages are for example

\[ L\{C,I,\neg,\cap,\cup,\Rightarrow\} \text{ or } L\{\Box,\Diamond,\neg,\cap,\cup,\Rightarrow\} \]
Some More Non-Extensional Connectives

Knowledge logics also extend the classical logic by adding a new one argument knowledge connective. The knowledge connective is often denoted by $K$.

A formula $KA$ reads: it is known that $A$ or $A$ is known.

A language of a knowledge logic is for example

$$\mathcal{L}\{K, \neg, \cap, \cup, \Rightarrow\}$$
Some More Non-Extensional Connectives

**Autoepistemic logics** extend classical logic by adding an one argument believe connective, often denoted by $B$
A formula $BA$ reads: *it is believed that A*
A language of an autoepistemic logic is for example

$$\mathcal{L}\{B, \neg, \cap, \cup, \Rightarrow\}$$
Some More Non-Extensional Connectives

**Temporal logics** also extend classical logic by adding one argument temporal connectives

Some of temporal connectives are: F, P, G, H.

Their intuitive meanings are:

- **FA** reads *A is true at some future time,*
- **PA** reads *A was true at some past time,*
- **GA** reads *A will be true at all future times,*
- **HA** reads *A has always been true in the past*
Propositional Connectives

It is possible to create connectives with more than one or two arguments.
We consider here only one or two argument connectives.
Chapter 3
Propositional Languages
PART 2: Formal Definitions
Propositional Language

Definition
A propositional language is a pair

\[ L = (\mathcal{A}, \mathcal{F}) \]

where \( \mathcal{A}, \mathcal{F} \) are called an alphabet and a set of formulas, respectively.

Definition
Alphabet is a set

\[ \mathcal{A} = \text{VAR} \cup \text{CON} \cup \text{PAR} \]

\text{VAR, CON, PAR} are all disjoint sets of propositional variables, connectives and parenthesis, respectively.

The sets \( \text{VAR}, \text{CON} \) are non-empty.
Alphabet Components

VAR is a countably infinite set of propositional variables. We denote elements of VAR by

\[ a, b, c, d, \ldots \]

with indices if necessary.

\( CON \neq \emptyset \) is a finite set of propositional connectives.

We assume that the set \( CON \) of logical connectives is non-empty, i.e. that a propositional language always has at least one connective.
Alphabet Components

Notation
We denote the language $L$ with the set of connectives $CON$ by

$L_{CON}$

Observe that propositional languages differ only on a choice of the propositional connectives, hence our notation.
Alphabet Components

PAR is a set of auxiliary symbols
This set may be empty; for example in case of Polish notation.

Assumptions
We assume here that PAR contains only 2 parenthesis and

\[ PAR = \{(,\}\} \]

We also assume that the set CON of logical connectives contains only unary and binary connectives, i.e.

\[ CON = C_1 \cup C_2 \]

where \( C_1 \) is the set of all unary connectives, and \( C_2 \) is the set of all binary connectives
Formulas Definition

Definition

The set $\mathcal{F}$ of all formulas of a propositional language $\mathcal{L}_{CON}$ is built recursively from the elements of the alphabet $\mathcal{A}$ as follows.

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the smallest set for which the following conditions are satisfied:

1. $\text{VAR} \subseteq \mathcal{F}$
2. If $A \in \mathcal{F}$, $\triangledown \in C_1$, then $\triangledown A \in \mathcal{F}$
3. If $A, B \in \mathcal{F}$, $\circ \in C_2$ i.e. $\circ$ is a two argument connective, then $(A \circ B) \in \mathcal{F}$

By (1) propositional variables are formulas and they are called atomic formulas.

The set $\mathcal{F}$ is also called a set of all well formed formulas (wff) of the language $\mathcal{L}_{CON}$.
Set of Formulas

**Observe** that the alphabet $\mathcal{A}$ is countably infinite. Hence the set $\mathcal{A}^*$ of all finite sequences of elements of $\mathcal{A}$ is also countably infinite. By definition $\mathcal{F} \subseteq \mathcal{A}^*$ and hence we get that the set of all formulas $\mathcal{F}$ is also countably infinite.

We state as separate fact:

**Fact**

For any propositional language $\mathcal{L} = (\mathcal{A}, \mathcal{F})$, its sets of formulas $\mathcal{F}$ is always a countably infinite set.

We hence consider here only infinitely countable languages.
Main Connectives and Direct Sub-Formulas

\( \nabla \) is called a main connective of the formula \( \nabla A \in \mathcal{F} \)

\( A \) is called its direct sub-formula of \( \nabla A \)

\( \circ \) is called a main connective of the formula \( (A \circ B) \in \mathcal{F} \)

\( A, B \) are called direct sub-formulas of \( (A \circ B) \)
Examples

E1  Main connective of \((a \Rightarrow \neg Nb)\) is \(\Rightarrow\)
a, \(\neg Nb\) are direct sub-formulas

E2  Main connective of \(N(a \Rightarrow \neg b)\) is \(N\)
\((a \Rightarrow \neg b)\) is the direct sub-formula

E3  Main connective of \(\neg(a \Rightarrow \neg b)\) is \(\neg\)
\((a \Rightarrow \neg b)\) is the direct sub-formula

E4  Main connective of of \((\neg a \cup \neg(a \Rightarrow b))\) is \(\cup\)
\(\neg a, \neg(a \Rightarrow b))\) are direct sub-formulas
Sub-Formulas

We define a notion of a sub-formula in two steps:

**Step 1**
For any formulas $A$ and $B$, the formula $A$ is a proper sub-formula of $B$ if there is a sequence of formulas, beginning with $A$, ending with $B$, and in which each term is a direct sub-formula of the next.

**Step 2**
A sub-formula of a given formula $A$ is any proper sub-formula of $A$, or $A$ itself.
Sub-Formulas Example

The formula \((\neg a \cup \neg(a \Rightarrow b))\)
has two direct sub-formulas: \(\neg a\), \(\neg(a \Rightarrow b)\)
The direct sub-formulas of \(\neg a\), \(\neg(a \Rightarrow b)\)
are respectively \(a\), \((a \Rightarrow b)\)
The direct sub-formulas of \(a\), \((a \Rightarrow b)\), are \(a, b\)
END of the process
Example

Given a formula

$$(\neg a \cup \neg(a \Rightarrow b))$$

Its set of all proper sub-formulas is:

$$S = \{\neg a, \neg(a \Rightarrow b), a, (a \Rightarrow b, )b\}$$

The set of all its sub-formulas is

$$S \cup \{(\neg a \cup \neg(a \Rightarrow b))\}$$
We define a degree of a formula as a number of occurrences of logical connectives in the formula.

Example

The degree of \((\neg a \cup \neg(a \Rightarrow b))\) is 4

The degree of \(\neg(a \Rightarrow b))\) is 2

The degree of \(\neg a\) is 1

The degree of \(a\) is 0
Formula Degree

A degree of a formula is number of occurrences of logical connectives in the formula

Observation: the degree of any proper sub-formula of $A$ must be one less than the degree of $A$

This is the central fact upon which mathematical induction arguments are based.

Proofs of properties of formulas are usually carried by mathematical induction on their degrees
Exercise

Exercise 1
Consider a language

\[ \mathcal{L} = \mathcal{L}\{\neg, \diamond, \Box, \cup, \cap, \Rightarrow\} \]

and a set \( S \subseteq \mathcal{A}^* \) such that

\[
S = \{\diamond \neg a \Rightarrow (a \cup b), (\diamond (\neg a \Rightarrow (a \cup b))), \\
\diamond \neg (a \Rightarrow (a \cup b))\}
\]

1. Determine which of the elements of \( S \) are, and which are not well formed formulas (wff) of \( \mathcal{L} \)
2. If a formula \( A \) is a well formed formula, i.e. \( A \in \mathcal{F} \), determine its main connective.
3. If \( A \notin \mathcal{F} \) write the correct formula and then determine its main connective
Solution
The formula $\Diamond \neg a \Rightarrow (a \cup b)$ is not a well formed formula.
The correct formula is

$$(\Diamond \neg a \Rightarrow (a \cup b))$$

The main connective is $\Rightarrow$
The correct formula says:
If negation of $a$ is possible, then we have $a$ or $b$

Another correct formula in is

$$\Diamond (\neg a \Rightarrow (a \cup b))$$

The main connective is $\Diamond$
The corrected formula says:
It is possible that not $a$ implies $a$ or $b$
Exercise 1 Solution

The formula \((\Diamond(\neg a \Rightarrow (a \cup b)))\) is not correct

The correct formula is

\[\Diamond(\neg a \Rightarrow (a \cup b))\]

The main connective is \(\Diamond\)

The correct formula says:
It is possible that not \(a\) implies \(a\) or \(b\)

\[\Diamond(\neg (a \Rightarrow (a \cup b)))\] is a correct formula

The main connective is \(\Diamond\)

The formula says:
It is possible that it is not true that \(a\) implies \(a\) or \(b\)
Exercise

Exercise 2
Given a formula:

\[ \Diamond((a \cup \neg a) \cap b) \]

1. Determine its degree
2. Write down all its sub-formulas

Solution:
The degree is 4
All sub-formulas are:

\[ \Diamond((a \cup \neg a) \cap b), ((a \cup \neg a) \cap b), \]

\[ (a \cup \neg a), \neg a, b, a \]
Language Defined by a set $S$

**Definition**
Given a set $S$ of formulas of a language $\mathcal{L}_{\text{CON}}$

Let $CS \subseteq \text{CON}$ be the set of all connectives that appear in formulas of $S$.

A language $\mathcal{L}_{CS}$ is called the language defined by the set of formulas $S$.

**Example**
Let $S$ be a set

$$S = \{ ((a \Rightarrow \neg b) \Rightarrow \neg a), \Box (\neg \Diamond a \Rightarrow \neg a) \}$$

All connectives appearing in the formulas in $S$ are:

$$\Rightarrow, \neg, \Box, \Diamond$$

The language defined by the set $S$ is

$$\mathcal{L}\{\neg, \Rightarrow, \Box, \Diamond\}$$
Exercise

Exercise 3
Write the following natural language statement:

*From the fact that it is possible that Anne is not a boy we deduce that it is not possible that Anne is not a boy or, if it is possible that Anne is not a boy, then it is not necessary that Anne is pretty*

in the following two ways

1. As a formula
   \[ A_1 \in \mathcal{F}_1 \] of a language \( \mathcal{L}\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\} \)

2. As a formula
   \[ A_2 \in \mathcal{F}_2 \] of a language \( \mathcal{L}\{\neg, \cap, \cup, \Rightarrow\} \)
Exercise 3 Solution

1. We translate our statement into a formula $A_1 \in \mathcal{F}_1$ of the language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$ as follows

Propositional Variables: $a, b$

$a$ denotes statement: *Anne is a boy*,

$b$ denotes a statement: *Anne is pretty*

Propositional Modal Connectives: $\Box, \Diamond$

$\Diamond$ denotes statement: *it is possible that*

$\Box$ denotes statement: *it is necessary that*

Translation 1: the formula $A_1$ is

$$(\Diamond \neg a \Rightarrow (\neg \Diamond \neg a \cup (\Diamond \neg a \Rightarrow \neg \Box b)))$$
Exercise 3 Solution

2. We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of the language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$ as follows

Propositional Variables: $a, b$

$a$ denotes statement: it is possible that Anne is not a boy

$b$ denotes a statement: it is necessary that Anne is pretty

Translation 2: the formula $A_2$ is

$$(a \Rightarrow (\neg a \lor (a \Rightarrow \neg b)))$$
Exercise

Exercise 4

Write the following natural language statement:

*For all natural numbers $n \in \mathbb{N}$ the following implication holds: if $n < 0$, then there is a natural number $m$, such that it is possible that $n + m < 0$, OR it is not possible that there is a natural number $m$, such that $m > 0*"

in the following two ways

1. As a formula

   $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}\{\neg, \cap, \cup, \Rightarrow\}$

2. As a formula

   $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}$
Exercise 4 Solution

1. We translate our statement into a formula $A_1 \in F_1$ of the language $\mathcal{L}\{\neg, \wedge, \vee, \Rightarrow\}$ as follows

Propositional Variables: $a$, $b$

$a$ denotes a statement:  *For all natural numbers $n \in N$ the following implication holds: if $n < 0$, then there is a natural number $m$, such that it is possible that $n + m < 0$*

$b$ denotes a statement:  *it is not possible that there is a natural number $m$, such that $m > 0$*

Translation: the formula $A_1$ is

$$(a \cup \neg b)$$
2. We translate our statement into a formula \( A_2 \in \mathcal{F}_2 \) of a language \( \mathcal{L}\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\} \) as follows

**Propositional Variables:** \( a, b \)

- \( a \) denotes a statement: *For all natural numbers \( n \in N \) the following implication holds: if \( n < 0 \), then there is a natural number \( m \), such that it is possible that \( n + m < 0 \*)

- \( b \) denotes a statement: *there is a natural number \( m \), such that \( m > 0 \*)

**Translation:** the formula \( A_2 \) is

\[
(a \cup \neg \Diamond b)
\]
Exercise

Exercise 5
Write the following natural language statement:

The following statement holds for all natural numbers \( n \in \mathbb{N} \):

if \( n < 0 \), then there is a natural number \( m \), such that it is possible that \( n + m < 0 \), OR it is not possible that there is a natural number \( m \), such that \( m > 0 \)

in the following two ways

1. As a formula
   \[ A_1 \in \mathcal{F}_1 \text{ of a language } \mathcal{L}\{\neg, \cap, \cup, \Rightarrow\} \]

2. As a formula
   \[ A_2 \in \mathcal{F}_2 \text{ of a language } \mathcal{L}\{\neg, \Box, \lozenge, \cap, \cup, \Rightarrow\} \]
Exercise

Exercise 6
Write the following natural language statement:

*From the fact that each natural number is greater than zero we deduce that it is not possible that Anne is a boy or, if it is possible that Anne is not a boy, then it is necessary that it is not true that each natural number is greater than zero*

in the following two ways

1. As a formula
   \[ A_1 \in F_1 \text{ of a language } L_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}} \]

2. As a formula
   \[ A_2 \in F_2 \text{ of a language } L_{\{\neg, \cap, \cup, \Rightarrow\}} \]