cse371/mat371
LOGIC

Professor Anita Wasilewska

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LECTURE 11

Chapter 11
Introduction to Intuitionistic Logic

Short History

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as intuitionism

Intuitionism was originated by L. E. J. Brouwer in 1908

The first Hilbert style formalization of the **Intuitionistic logic** formulated as a proof system only, is due to A. Heyting in 1930

We present here a Hilbert style proof system **I** for **Intuitionistic Propositional Logic**

The proof system **I** is equivalent to the Heyting's original formalization

We also discuss a **relationship** between the Intuitionistic and Classical logics

Short History

There have been, of course, several successful attempts at creating **semantics** for the intuitionistic logic, and hence to define formally a notion of the **intuitionistic tautology**

The most known are Kripke models and algebraic models

Kripke models were defined by Kripke in 1964

Algebraic models were initiated by Stone and Tarski in 1937, 1938, respectively

An uniform theory and presentation of **topological** and **algebraic models** was given by Rasiowa and Sikorski in 1964

Hilbert Proof System for Intuitionistic Propositional Logic

Language

We adopt a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}}$$

with the set of formulas denoted by $\mathcal{F}$

Logical Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

Hilbert Proof System for Intuitionistic Propositional Logic

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

where A, B, C are any formulas in $\mathcal{L}$

Rules of inference

We adopt a Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only rule of inference

Proof System I

A proof system

$$\mathbf{I} = (\ \mathcal{L}_{\{\neg,\ \cup,\ \cap,\ \Rightarrow\}},\ \mathcal{F},\ \ \{A1, ..., A11\},\ (MP)\ )$$

is called a Hilbert Style Formalization for Intuitionistic Propositional Logic

The set of axioms $\{A1, ..., A11\}$ is due to Rasiowa (1959)

It differs from Heyting's original set of axioms but they are equivalent

We introduce, as usual, the notion of a formal proof in **I** and denote by

$$\vdash_I A$$

the fact that a formula A has a formal proof in **I** and we say that the formula A is **intuitionistically provable**

Completeness Theorem

There are several ways one can define a **semantics** for the intuitionistic logic

Define a semantics for the intuitionistic logic means to define the semantics for the original Heyting proof system and **prove the Completeness Theorem** for it under this semantics

The same applies to any other equivalent proof system, in particular for our proof system **I**

Completeness Theorem

The notion of intuitionistic semantics and hence the formal definition of **intuitionistic tautology** will be defined and discussed later

For a moment we denote by

$$\models_I A$$

the fact that A is an **intuitionistic tautology** under some **intuitionistic semantics**

Let's denote by **IS** any proof system **equivalent** to the original Heyting system for Intuitionistic logic

Completeness Theorem for the proof system **IS**
For any formula $A \in \mathcal{F}$,

$$\vdash_{IS} A \quad \text{if and only if} \quad \models_I A$$

Examples of Intuitionistic Tautologies

Of course, all of **Logical Axioms** A1 - A11 of our proof system **I** are Intuitionistic tautologies

Here are some other **classical tautologies** that are also **Intuitionistic tautologies**

1. $(A \Rightarrow A)$

2. $(A \Rightarrow (B \Rightarrow A))$

3. $(A \Rightarrow (B \Rightarrow (A \cap B)))$

4. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

5. $(A \Rightarrow \neg\neg A)$

6. $\neg(A \cap \neg A)$

7. $((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

Examples of Intuitionistic Tautologies

8. $(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$

9. $((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$

10. $((\neg A \cup \neg B) \Rightarrow \neg(A \cap B))$

11. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

12. $((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$

13. $(\neg\neg\neg A \Rightarrow \neg A)$

14. $(\neg A \Rightarrow \neg\neg\neg A)$

15. $(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$

16. $((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$

Examples of NOT Intuitionistic Tautologies

The following classical tautologies **are not** intuitionistic tautologies

17. $(A \cup \neg A)$

18. $(\neg\neg A \Rightarrow A)$

19. $((A \Rightarrow B) \Rightarrow (\neg A \cup B))$

20. $(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$

21. $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$

22. $((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$

23. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$,

Homework Exercises

The general idea of **algebraic models** for the intuitionistic logic is defined in terms of **Pseudo-Boolean Algebras** in the following way

A formula A is said to be an **intuitionistic tautology** if and only if $v \models A$, for all v and **all Pseudo-Boolean Algebras**, where v maps the propositional variable VAR into the universe of a **Pseudo-Boolean Algebra**

Definition

A formula A is an **intuitionistic tautology** if and only if it is true in all **Pseudo-Boolean Algebras** under all possible variable assignments v

Homework Exercises

The 3 element Heyting algebra **H** as defined in the section "Some three valued logics" is an example of a 3 element **Pseudo-Boolean Algebra**

Exercise 1

Show that the 3 element Heyting algebra **H** is a model for all logical axioms **A1- A11** and all of the formulas **1-16**, i.e. show that they are all **H- tautologies**

Exercise 2

Find for which of the formulas **17 - 23** the 3 element Heyting algebra acts as a **counter-model**

Connection Between Classical and Intuitionistic Logics

The first connection is quite obvious.

It was proved by Rasiowa and Sikorski in 1964 that by adding the axiom

A12 $(A \cup \neg A)$
to the set of axioms of our system **I** we obtain a Hilbert proof system **C** that is **complete** with respect to classical semantics

This proves the following.

Theorem 1
Every formula that is intuitionistically derivable is also classically derivable, i.e. the implication

$$\text{If} \quad \vdash_I A \quad \text{then} \quad \vdash_C A$$

holds for any $A \in \mathcal{F}$

Connection Between Classical and Intuitionistic Logics

We write

$$\models A$$

and

$$\models_I A$$

to denote that A is a **classical** and **intuitionistic** tautology, respectively.

As both proof systems **I** and **C** are complete under respective semantics, we can re-write Theorem 1 as the following **relationship** between classical and intuitionistic **tautologies**

Theorem 2 For any formula $A \in \mathcal{F}$,

$$\text{If } \models_I A, \text{ then } \models A$$

Connection Between Classical and Intuitionistic Logics

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa

The following has been proved by Glivenko in 1929 in terms of provability as the semantics for Intuitionisctic Logic didn't yet exist

Theorem 3 (Glivenko)

For any formula $A \in \mathcal{F}$,

A is **classically** provable if and only if $\neg\neg A$ is an **intuitionistically** provable, i.e.

$$\vdash_C A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol $\vdash_C$ for classical provability in a complete classical proof system

Connection Between Classical and Intuitionistic Logics

The following has been proved by Tarski in 1938 together with a **definition of algebraic semantics** for Intuitionistic Logic

Theorem 4 (Tarski)

For any formula $A \in \mathcal{F}$,

A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \quad \text{if and only if} \quad \models_I \neg\neg A$$

Connection Between Classical and Intuitionistic Logics

The following relationships were proved by Gödel in 1331.

Theorem 5 (Gödel)

For any formulas $A, B \in \mathcal{F}$,

a formula $(A \Rightarrow \neg B)$ is **classically provable** if and only if it is **intuitionistically provable**, i.e.

$$\vdash_C (A \Rightarrow \neg B) \quad \text{if and only if} \quad \vdash_I (A \Rightarrow \neg B)$$

Theorem 6 (Gödel)

For any formula $A, B \in \mathcal{F}$,

If A contains no connectives except $\cap$ and $\neg$, then A i is **classically provable** if and only if it is **intuitionistically provable**

Connection Between Classical and Intuitionistic Logics

By the **Completeness Theorems** for classical and intuitionisctic logics we get the following equivalent **semantic** form of Gödel' s **Theorems 5, 6**

Theorem 6

A formula $(A \Rightarrow \neg B)$ is a **classical tautology** if and only if it is an **intuitionistic tautology**, i.e.

$$\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)$$

Theorem 7

If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology

On intuitionistically derivable disjunction

In a **classical logic** it is possible for the disjunction $(A \cup B)$ to be a **tautology** when neither A nor B is a **tautology**

The tautology $(A \cup \neg A)$ is the simplest example

This does not hold for the **intuitionistic logic**

This fact was stated without the proof by Gödel in 1931 and **proved** by Gentzen in 1935 via his proof system **LI** which is presented and discussed in chapter 12 and Lecture 15

On intuitionistically derivable disjunction

Remember that Gödel and Gentzen meant by **intuitionistic logic** a Heyting proof system or any other proof system (like the one defined by Gentzen) equivalent with it

The following theorem was announced without the proof by Gödel in 1931 and proved by Gentzen in 1934

Theorem 8 (Gödel, Gentzen)

A disjunction $(A \cup B)$ is **intuitionistically provable** if and only if either A or B is **intuitionistically provable** i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \quad \text{or} \quad \vdash_I B$$

We obtain, via the **Completeness Theorem** the following equivalent semantic version of the above

Theorem 9

A disjunction $(A \cup B)$ is **intuitionistic tautology** if and only if either A or B is **intuitionistic tautology**, i.e.

$$\models_I (A \cup B) \quad \text{if and only if} \quad \models_I A \quad \text{or} \quad \models_I B$$

Chapter 11
Gentzen System **LI** for Intuitionistic Logic

Gentzen System **LI** for Intuitionistic Logic

Definition of Gentzen System **LI**

The proof system **LI** for Intuitionistic Logic as presented here was published by **G. Gentzen** in 1935

It was presented as a particular case of his proof system **LK** for the classical logic

We present now the **original Gentzen** proof system **LI** and then we show how it can be extended to the **original Gentzen** system **LK**

Language of **LI**

Language of **LI** is

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

We add a new symbol $\longrightarrow$ to the language and call it a Gentzen arrow

We **denote**, as before, the finite sequences of formulas by Greek capital letters

$$\Gamma, \Delta, \Sigma, \ldots$$

with indices if necessary

Language of **LI**

Definition Any expression

$$\Gamma \longrightarrow \Delta$$

where $\Gamma, \Delta \in \mathcal{F}^*$ and

Δ consists of **at most one formula**

is called a **LI** sequent

We denote the set of all **LI** sequents by *ISQ*, i.e.

$ISQ = \{\Gamma \longrightarrow \Delta :$ Δ consists of **at most one formula**$\}$

Axioms of **LI**

Axioms of LI consist of any sequent from the set *ISQ* which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma,\ A,\ \Delta\ \longrightarrow\ A$$

for $\Gamma, \Delta \in \mathcal{F}^*$

Rules of Inference of **LI**

The set inference rules of **LI** is divided into two groups : the **structural rules** and the **logical rules**

There are three **Structural Rules** of **LI**: Weakening, Contraction and Exchange

Weakening structural rule

$$(weak \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

$$(\rightarrow weak) \quad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow A}$$

A is called the weakening formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Contraction structural rule

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta}$$

The case below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A}{\Gamma \longrightarrow \Delta, A}$$

A is called the contraction formula

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Exchange structural rule

$$(exch \rightarrow) \quad \frac{\Gamma_1,\ A, B,\ \Gamma_2 \ \longrightarrow\ \Delta}{\Gamma_1,\ B, A,\ \Gamma_2 \ \longrightarrow\ \Delta}$$

The case below is **not VALID** for **LI**; we list it as it will be used in the classical case

$$(\rightarrow exch) \quad \frac{\Delta \longrightarrow \Gamma_1,\ A, B,\ \Gamma_2}{\Delta \longrightarrow \Gamma_1,\ B, A,\ \Gamma_2}.$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Logical Rules

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)}$$

$$(\rightarrow \cup)_2 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)}$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Rules of Inference of **LI**

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)}$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \quad ; \quad B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta}$$

Remember that Δ contains at most one formula

Gentzen System **LI**

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow}$$

$$(\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}$$

We define the Gentzen System **LI** as

$$\mathbf{LI} = (\mathcal{L}, \ ISQ, \ AX, \ \text{Structural rules}, \ \text{Logical rules})$$

Gentzen Sequent Calculus **LI** for Intuitionistic Logic
Part 2

Decomposition Trees in **LI**

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL**

In all systems the proof search procedure consists of building the **decomposition trees**

Remark 1

In **RS** the decomposition tree $\mathbf{T}_A$ of any formula A is always unique

Remark 2

In **GL** the "blind search" defines, for any formula A a **finite number** of decomposition trees,

Nevertheless, it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules

Decomposition Trees in **LI**

Remark 3

In **LI** the structural rules play a **vital role** in the proof construction and hence, in the proof search

The fact that a given **decomposition tree** ends with an non-axiom leaf **does not always imply** that does not exist

It might only imply that our search strategy was **not good**

The problem of **deciding** whether a given formula A **does, or does not** have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic

Examples

Example 1

Determine] whether

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Observe that

If we find a decomposition tree of A in **LI** such that all its leaves are axiom, we have a proof , i.e

$$\vdash_{\mathbf{LI}} A$$

If all possible decomposition trees have a non-axiom leaf then the proof of A i n **LI** does not exist, i.e.

$$\nvdash_{\mathbf{LI}} A$$

Examples

Consider the following decomposition tree $\mathbf{T1_A}$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg (A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg (A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge (\cup \longrightarrow)$$

$A \longrightarrow B$ $\qquad\qquad$ $B \longrightarrow B$

non – axiom $\qquad\qquad$ *axiom*

Examples

The tree **$T1_A$** has a non-axiom leaf, so it does not constitute a proof in **LI**

Observe that the decomposition tree in **LI** is not always unique

Hence this fact does not yet prove that a **proof** of A does not exist

Consider the following decomposition tree **$T2_A$**

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$A, \neg A, \neg B \longrightarrow$	$B, \neg A, \neg B \longrightarrow$
$\mid (exch \longrightarrow)$	$\mid (exch \longrightarrow)$
$\neg A, A, \neg B \longrightarrow$	$B, \neg B, \neg A \longrightarrow$
$\mid (\neg \longrightarrow)$	$\mid (exch \longrightarrow)$
$A, \neg B \longrightarrow A$	$\neg B, B, \neg A \longrightarrow$
axiom	$\mid (\neg \longrightarrow)$
	$B, \neg A \longrightarrow B$; *axiom*

Examples

All leaves of $\mathbf{T2_A}$ are axioms and hence $\mathbf{T2_A}$ is a **a proof** in **LI**

Hence we proved that

$$\vdash_{\mathbf{LI}} ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$$

Examples

Example 2: Show that

1. $\vdash_{\mathbf{LI}} (A \Rightarrow \neg\neg A)$

2. $\nvdash_{\mathbf{LI}} (\neg\neg A \Rightarrow A)$

Solution of **1.**

We construct some, or all decomposition trees of

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

The tree $\mathbf{T}_A$ that ends with all axioms leaves is a proof of A in **LI**

Examples

We construct $\mathbf{T}_A$ as follows

$$\longrightarrow (A \Rightarrow \neg\neg A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$A \longrightarrow \neg\neg A$$

$$\mid (\longrightarrow \neg)$$

$$\neg A, A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow A$$

axiom

All leaves of $\mathbf{T}_A$ are axioms what proves that we have found a proof

We **don't need** to construct any other decomposition trees.

Examples

Solution of **2.**

In order to prove that

$$\nvdash_{\mathbf{LI}} \quad (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

and show that each of them has an non-axiom leaf

Examples

Here is **$\mathbf{T1}_A$**

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

one of 2 choices

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

one of 3 choices

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

one of 3 choices

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

one of 2 choices

$$A \longrightarrow$$

non – axiom

Here is $\mathbf{T2}_A$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$\mid (\longrightarrow\Rightarrow)$ *one of 2 choices*

$$\neg\neg A \longrightarrow A$$

$\mid (contr \longrightarrow)$ *second of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$\mid (\longrightarrow weak)$ *first of 2 choices*

$$\neg\neg A, \neg\neg A \longrightarrow$$

$\mid (\neg \longrightarrow)$ *first of 2 choices*

$$\neg\neg A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *one of 2 choices*

$$A, \neg\neg A \longrightarrow$$

$\mid (exch \longrightarrow)$ *one of 2 choices*

$$\neg\neg A, A \longrightarrow$$

$\mid (\neg \longrightarrow)$*one of 2 choices*

$$A \longrightarrow \neg A$$

$\mid (\longrightarrow \neg)$ *first of 2 choices*

$$A, A \longrightarrow$$

non – axiom

Structural Rules

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more **complicated trees**

This is due to the presence of structural rules "blind" application of the rule $(contr \rightarrow)$ gives always an infinite number of decomposition trees

In order to decide that none of them will produce a proof we need some **extra knowledge** about patterns of their construction, or just simply about the number o useful of application of **structural rules** within the proofs.

Structural Rules

In this case we can just make an ”**external**” **observation** that the our first tree **T1**$_A$ is in a sense a minimal one

It means that all other trees would only **complicate** this one in an inessential way, i.e. the we will never produce a tree with all axioms leaves

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its **correctness** is needed and that requires some extra knowledge

Within the scope of this book we accept the ”external explanation as a **sufficient solution**, provided its correctness had been proved elsewere

Structural Rules

As we can see from the above examples the structural rules and especially the $(contr \longrightarrow)$ rule **complicates** the proof searching task.

Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules

They also are as we have proved, **complete** with respect to classical semantics.

The original Gentzen system **LK** which does contain the structural rules is also, as proved by Gentzen, **complete**

Structural Rules

Hence all three classical proof system **RS, GL, LK** are equivalent

This proves that the structural rules can be **eliminated** from the system **LK**

A natural question of elimination of structural rules from the Intutionistic Gentzen system **LI** arises

The following **example** illustrates the negative answer

Connection Between Classical and Intuitionistic Logics

Here is the **connection** between Intuitionistic logic and the Classical one

Theorem 1

For any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where

$\models A$ means that A is a **classical tautology**

$\vdash_{IS} A$ means that A is Intutionistically provable in any **Intuitionistically complete** proof system IS

Connection Between Classical and Intuitionistic Logics

A Gentzen system **LI** has been proved to be **Intuitionistically complete** so have that the following

Theorem 2

For any formula $A \in \mathcal{F}$,

$$\models A \quad \text{if and only if} \quad \vdash_{\mathbf{LI}} \neg\neg A$$

Example

Example 3

Obviously

$$\models (\neg\neg A \Rightarrow A)$$

so by **Theorem 2** we must have that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg\neg A \Rightarrow A)$$

We are going to prove now that the structural rule $(contr \longrightarrow)$ is **essential** to the existence of the proof, i.e

We show now that the formula We $\neg\neg(\neg\neg A \Rightarrow A)$ is **not provable** in **LI** without the rule $(contr \longrightarrow)$

The following decomposition tree $\mathbf{T}_A$ is a proof of $A = \neg\neg(\neg\neg A \Rightarrow A)$ in **LI** with use of the contraction rule $(contr \longrightarrow)$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg\neg A, A \longrightarrow A$$

axiom

Contraction Rule

Assume now that the Contraction rule $(contr \longrightarrow)$ is **not available**

All possible decomposition trees are as follows

Tree **T1**$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$A \longrightarrow$$

$$non-axiom$$

Contraction Rule

The next is $\mathbf{T2}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non - axiom$$

Contraction Rule

The next is $\mathbf{T3}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

$$non - axiom$$

Contraction Rule

The last one is **T4**$_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$]$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow weak)$$

$$\longrightarrow$$

non – axiom

Contraction Rule

We have considered all possible decomposition trees that do not involve the Contraction Rule and **none** of them was a proof

This shows that the formula

$$\neg\neg(\neg\neg A \Rightarrow A)$$

is not provable in **LI** without $(contr \longrightarrow)$ rule, i.e. that

Fact

The Contraction Rule **can't be eliminated** from **LI**