# CSE/MAT371 QUIZ 7 SOLUTIONS Fall 2016 (20pts)

**QUESTION 1** (5pts)

Let **GL** be the Gentzen style proof system for classical logic.

Prove, by constructing a proper decomposition tree that

$\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

**Solution** THIS IS NOT THE ONLY SOLUTION!

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge(\Rightarrow\rightarrow$$

| | |
|---|---|
| $b, a \longrightarrow \neg(a \cap b), b$ | $b, a, b \longrightarrow b$ |
| $\mid (\rightarrow \neg)$ | *axiom* |
| $b, a, (a \cap b) \longrightarrow b$ | |
| $\mid (\cap \rightarrow)$ | |
| $b, a, a, b \longrightarrow b$ | |
| *axiom* | |

All leaves of the decomposition tree are axioms, hence the proof has been found.

**QUESTION 2** (10pts)

We know that **GL** is **strongly sound**, use a decomposition tree $\mathbf{T}_{\rightarrow A}$ to construct a **counter model** for a formula $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

**Solution** This is not the only correct Tree!

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\rightarrow$$

$$\longrightarrow a, b, a \qquad\qquad b \longrightarrow b, a$$

$$non - axiom \qquad\qquad axiom$$

The **counter-model** determined by $\mathbf{T}_{\rightarrow A}$ is any truth assignment v that evaluates the non axiom leaf $\longrightarrow b, b, a$ to F.

By the **strong soundness**, the value F "climbs the tree" and we get that also $v * (A) = F$.

We evaluate $v^*(\longrightarrow b, b, a) = (T \Rightarrow v(b) \cup v(b) \cup v(a)) = F$ if and only if $v(b) = v(a) = F$.

The **counter model** determined by the tree $\mathbf{T}_{\rightarrow A}$ is any $v : VAR \longrightarrow \{T, F\}$ such that $v(b) = v(a) = F$

**QUESTION 3** (5pts)

Use the **completeness theorem** for **GL** to prove that $\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

**Solution**

By the **Completeness Theorem** we have that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad textif\ \ and\ only\ \ if \quad \not\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Any v, such that $v(a) = v(b) = F$ is a counter-model for $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$, hence By the **Completeness Theorem** $\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

**GL Proof System**

**Axioms of GL**

$$\Gamma'_1, a, \Gamma'_2 \;\longrightarrow\; \Delta'_1, a, \Delta'_2$$

for any $a \in VAR$ and any sequences $\Gamma'_1, \Gamma'_2, \Delta'_1, \Delta'_2 \in VAR^*$

**Inference rules of GL**
**Conjunction rules**

$$(\cap \rightarrow)\ \frac{\Gamma', A, B, \Gamma \;\longrightarrow\; \Delta'}{\Gamma', (A \cap B), \Gamma \;\longrightarrow\; \Delta'} \qquad (\rightarrow \cap)\ \frac{\Gamma \;\longrightarrow\; \Delta, A, \Delta' \;;\; \Gamma \;\longrightarrow\; \Delta, B, \Delta'}{\Gamma \;\longrightarrow\; \Delta, (A \cap B), \Delta'}$$

**Disjunction rules**

$$(\rightarrow \cup)\ \frac{\Gamma \;\longrightarrow\; \Delta, A, B, \Delta'}{\Gamma \;\longrightarrow\; \Delta, (A \cup B), \Delta'} \qquad (\cup \rightarrow)\ \frac{\Gamma', A, \Gamma \;\longrightarrow\; \Delta' \;;\; \Gamma', B, \Gamma \;\longrightarrow\; \Delta'}{\Gamma', (A \cup B), \Gamma \;\longrightarrow\; \Delta'}$$

**Implication rules**

$$(\rightarrow\Rightarrow)\ \frac{\Gamma', A, \Gamma \;\longrightarrow\; \Delta, B, \Delta'}{\Gamma', \Gamma \;\longrightarrow\; \Delta, (A \Rightarrow B), \Delta'} \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma', \Gamma \;\longrightarrow\; \Delta, A, \Delta' \;;\; \Gamma', B, \Gamma \;\longrightarrow\; \Delta, \Delta'}{\Gamma', (A \Rightarrow B), \Gamma \;\longrightarrow\; \Delta, \Delta'}$$

**Negation rules**

$$(\neg \rightarrow)\ \frac{\Gamma', \Gamma \;\longrightarrow\; \Delta, A, \Delta'}{\Gamma', \neg A, \Gamma \;\longrightarrow\; \Delta, \Delta'} \qquad (\rightarrow \neg)\ \frac{\Gamma', A, \Gamma \;\longrightarrow\; \Delta, \Delta'}{\Gamma', \Gamma \;\longrightarrow\; \Delta, \neg A, \Delta'}$$