CSE/MAT371 QUIZ 6 SOLUTIONS Fall 2016

QUESTION 1 Consider a **strongly sound** system **RS1** obtained from **RS** by **changing** the sequence Γ' into Γ in all of the rules of inference of **RS**.

1. Construct **all** decomposition trees of a formula A: $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

2. Use one of your trees to define a counter model for A determined by the tree

Solution 1. Here it decomposition tree **T1** with the possible decomposition choices marked and chosen.

$$\mathbf{T1}_A$$

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

one choice

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ (\neg b \Rightarrow a)$$

two choices : **first formula choice**

$$\bigwedge (\neg \Rightarrow$$

$a,\ (\neg b \Rightarrow a)$	$\neg b,\ (\neg b \Rightarrow a$
one choice	*one choice*
$\mid (\Rightarrow)$	$\mid (\Rightarrow)$
$a,\ \neg\neg b, a$	$\neg b,\ \neg\neg b, a$
one choice	*one choice*
$\mid (\neg\neg)$	$\mid (\neg\neg$
$a,\ b, a$	$\neg b,\ b, a$
non axiom	*axiom*

The tree contains a **non- axiom** leaf, hence it is **not a proof**.

Solution 2. The system is **strongly sound**, so it is enough to find a counter model for a non axiom leaf as in strong sound systems ”F climbs” the tree and the leaf counter model is also a counter model for all sequences on the branch that ends with this leaf. In particular it is a counter model for the the root of the tree, i.e. the formula A.

The **counter model** for the leaf $a,\ b, a$ and hence for the **formula** A is any $v : VAR \longrightarrow \{T, F\}$ such that $v(a) = v(b) = F$.

Solution 1. Here it decomposition tree **T2** with the possible decomposition choices marked and chosen.

$$\mathbf{T2}_A$$

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\textit{one choice}$$

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ (\neg b \Rightarrow a)$$

$$\textbf{second formula choice}$$

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ \neg\neg b, a$$

$$\textit{two choices}:\ \textbf{first formula choice}$$

$$\bigwedge(\neg \Rightarrow$$

$$a,\ \neg\neg b, a) \qquad \neg b,\ \neg\neg b, a)$$

$$\mid (\neg\neg) \qquad \mid (\neg\neg)$$

$$a,\ b, a \qquad \neg b,\ b, a$$

$$\textit{non axiom} \qquad \textit{axiom}$$

The tree contains a **non- axiom** leaf, hence it is **not a proof**.

We have explored the two of first two choices and one of the second second choices, so the only choice now is the second formula of the second two choices. It is out tree **T3**.

Solution 1. Here it decomposition tree **T3**.

$$\mathbf{T2}_A$$

$$((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ (\neg b \Rightarrow a)$$

$$\mid (\Rightarrow)$$

$$\neg(a \Rightarrow b),\ \neg\neg b, a$$

$$\textbf{second formula choice}$$

$$\mid (\neg\neg)$$

$$\neg(a \Rightarrow b),\ b, a$$

$$\bigwedge(\neg \Rightarrow$$

$$a,\ b, a \qquad \neg b,\ b, a$$

$$\textit{non axiom} \qquad \textit{axiom}$$

We have explored all choices. All possible trees are **T1, T2, T3**.

QUESTION 2 Consider a **strongly sound** system **RS2** obtained from **RS** by changing the sequence Γ' into Γ and Δ into Δ' in all of the rules of inference of **RS**.

1. Define in your own words, for any A, the decomposition tree $\mathbf{T}_A$ in **RS2**.

Solution The definition of the decomposition tree $\mathbf{T}_A$ is in its essence similar to the one for **RS**, except for the changes which reflect the **difference** in the corresponding rules of decomposition. It means now given a node Γ on a tree, we traverse it from **right** to **left** and **find** the first decomposable formula. The tree $\mathbf{T}_A$ is, as in the case of **RS** uniquely determined by the formula A.

We follow now the following steps

Step 1 Decompose A using a rule defined by its main connective.

Step 2 Traverse resulting sequence Γ on the new node of the tree from **right** to **left** and **find** the first decomposable formula.

Step 3 Repeat **Step 1** and **Step 2** until no more decomposable formulas

End of Tree Construction

2. Prove Completeness Theorem for **RS2**.

Solution

We know that **RS2** is strongly sound, so we have to prove only the completeness part of the **Completeness Theorem**. We prove the opposite implication

$$\text{If } \nvdash_{\mathbf{RS2}} A \text{ then } \not\models A.$$

Assume that A is any formula is such that $\nvdash_{\mathbf{RS2}} A$. The unique $\mathbf{T}_A$ contains a non-axiom leaf L_A. It **defines** a truth assignment v which **falsifies** it as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if a does not appear in } L_A \end{cases}$$

RS2 is **strongly sound**, so it is enough to find a counter model for a non axiom leaf as in strong sound systems ”F climbs” the tree and the leaf counter model is also a counter model for all sequences on the branch that ends with this leaf. In particular it is a counter model for the the root of the tree, i.e. the formula A, i.e. we proved that

$$\not\models A$$