QUESTION 1  We define, for $A, b_1, b_2, ..., b_n$ and truth assignment $v$ a corresponding formulas $A'$, $B_1, B_2, ..., B_n$ as follows: $A' = \begin{cases} A & \text{if } v'(A) = T \\ \neg A & \text{if } v'(A) = F \end{cases}$ and $B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$

We proved the following Main Lemma: For any formula $A = A(b_1, b_2, ..., b_n)$ and any truth assignment $v$, if $A'$, $B_1$, $B_2$, ..., $B_n$ are corresponding formulas defined above, then $B_1, B_2, ..., B_n \vdash A'$.

Let $A$ be a formula $((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$, and let $v$ be such that $v(a) = T$, $v(b) = F$.

Write what Main Lemma asserts for the formula $A$.

Solution  Observe that the formula $A$ is a basic tautology, hence $A' = A$.

$A = A(a, b)$ and we get $B_1 = a$, $B_2 = \neg b$ and Main Lemma asserts $a, \neg b \vdash ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$.

QUESTION 2  Consider the Hilbert system $H_2 = (\mathcal{L}_{\rightarrow}, \mathcal{F}, \{A1, A2\}, (MP) \frac{A \Rightarrow (B \Rightarrow A)}{B} )$, where $A1 : (A \Rightarrow (B \Rightarrow A))$, $A2 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

We know that the Main Lemma and following formula: $(*) : ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ are provable in $H_2$.

Part 1.  Explain why the Deduction Theorem holds for $H_2$.

Solution  Only axioms A1 and A2 were used in the proof of the Deduction Theorem, so its proof is valid in $H_2$.

Part 2.  The proof of Completeness Theorem defines a method of efficiently combining $v \in V_A$ while constructing the proof of a formula $A$.

Write all steps of the Proof 1 as applied to $A : ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$.

Long Solution  The completeness part of the Completeness Theorem states: if $\models A$, then $\vdash_{H_2} A$.

The formula $A : ((\neg a \Rightarrow \neg b) \Rightarrow (b \Rightarrow a))$ is a tautology, so the assumption $\models A$ holds and we CAN follow the proof. Also by Part 1, the Deduction theorem holds for $H_2$ and we know that the Main Lemma also holds and the formula: $(*) : ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ is provable in $H_2$.

By the Main Lemma and the assumption that $\models A(a, b)$ any $v \in V_A$ defines formulas $B_a$, $B_b$ such that $B_a, B_b \vdash A$.

The proof is based on a method of using all $v \in V_A$ (there is 4 of them) to carry a process of elimination of the hypothesis $B_a$, $B_b$ that constructs a formal proof of $A$, i.e. to prove that $\vdash A$.

Elimination of $B_b$.

We have to cases: $v(b) = T$ or $v(b) = F$.

Let $v(b) = T$, by definition $B_b = b$ and by the Main Lemma, $B_a, b \vdash_{H_2} A$, by Deduction Theorem we get $B_a \vdash_{H_2} (b \Rightarrow A)$.

Let $v(b) = F$, by definition, $B_b = \neg b$ and by the Main Lemma, $B_a, \neg b \vdash_{H_2} A$, by Deduction Theorem we get $B_a \vdash_{H_2} (\neg b \Rightarrow A)$.

Now we re-write the formula $(*)$ for $A = b, B = A$ and get that $\vdash_{H_2} ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$. By monotonicity, $B_a \vdash_{H_2} ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A))$.

We apply MP twice and get $B_a \vdash_{H_2} A$.

We eliminated $B_b$. We repeat the same process for $B_a$ as follows.
Elimination of \( B_a \).

We have again to cases: \( v(a) = T \) or \( v(a) = F \).
Let \( v(a) = T \), by definition \( B_a = a \) and by the Main Lemma, \( a \vdash H \), by Deduction Theorem we get \( \vdash H \) \( (a \Rightarrow A) \).
Let \( v(a) = F \), by definition, \( B_a = \neg a \) and by the Main Lemma, \( \neg a \vdash H \), by Deduction Theorem we get \( \vdash H \) \( (\neg a \Rightarrow A) \).
Now we re-write the formula \( * \) for \( A = a \), \( B = A \) and get that \( \vdash H \) \( ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)) \). we apply MP twice and get \( \vdash H \) \( A \).
This ends the proof.