CSE371 MIDTERM SOLUTIONS Fall 2015

PART 1: DEFINITIONS As in Lectures

PART 2: PROBLEMS

PROBLEM 1

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$, where **L** A represents statement "one likes A", "A is liked".

Solution We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b
a denotes statement: *play bridge*,
b denotes a statement: *the weather is good*

Translation 1

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c
a denotes statement: *One likes to play bridge* ,
b denotes a statement: *the weather is good*, and
c denotes a statement: *one likes not to play bridge*

Translation 2:

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

Problem 2

CREATE YOUR OWN 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ by **defining the connectives** $\neg$, $\cup$, $\Rightarrow$ on a set $\{F, \perp, T\}$ of logical values.

You must follow the following **assumptions**

Assumption 1

The third logical value value is **intermediate** between truth and falsity, i.e. the set of logical values is **ordered** and we have the following $F < \perp < T$

Assumption 2 T is the **designated value**

2. The semantics has to **model the situation** in which one "likes" only truth; i.e. in which

$\mathbf{L}T = T$ and $\mathbf{L} \perp = F$, $\mathbf{L}F = F$

3. The connectives $\neg$, $\cup$, $\Rightarrow$ can be defined as you wish, but you have to define them in such a way to make sure that

$$\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$$

Part 1 Write down definition of logical connectives

Solution

Here is MY **M** semantics - yours can be different!

I define the logical connectives by "shorthand" writing functions defining connectives in form of the "truth tables" and skipping other points of the definition - as I have typed it so many times for you before!

L Connective

$\mathbf{L}$	F	$\perp$	T
	F	F	T

Negation :

$\neg$	F	$\perp$	T
	T	F	F

Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	F	F	T

Disjunction :

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	T	T

Part 2

Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg \mathbf{L}A)$ under your semantics - you can use shorthand notation

Solution

We verify

$\mathbf{L}T \cup \neg \mathbf{L}T = T \cup F = T$, $\mathbf{L} \perp \cup \neg \mathbf{L} \perp = F \cup \neg F = F \cup T = T$, $\mathbf{L}F \cup \neg \mathbf{L}F = F \cup \neg F = T$

PROBLEM 3

Part 1 Verify whether the formulas $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$ and $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$ have a model/ counter model under your semantics **M**. You can use **shorthand notation**.

Solution

Given $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$, and

$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$

Any v, such that $v(a) = T$ is a **M model** for A_1 and for A_2 directly from the definition of $\cup$.

Part 2 Verify whether the following set $\mathbf{G}$ is $\mathbf{M}$-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{\ \mathbf{L}a,\ \ (a \cup \neg \mathbf{L}b),\ \ (a \Rightarrow b),\ b\ \}$$

Solution

Any v, such that $v(a) = T,\ v(b) = T$ is a **M model** for **G** as

$\mathbf{L}T = T, \quad (T \cup \neg \mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T$

PROBLEM 4

Let S be the following **proof system**

$$S = (\ \mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}},\ \mathcal{F},\ \ \{\mathbf{A1}, \mathbf{A2}\},\ \ \{r1,\ r2\}\ )$$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg \mathbf{L}A)$

A2 $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1)\ \frac{A\ ;B}{(A \cup B)}, \qquad\qquad (r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

Part 1

Show, by constructing a proper **formal proof** that

$$\vdash_S\ ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$$

You must write comments how each step pot the proof was obtained

Write all steps of the **formal proof** as follows- write as MANY as you NEED!

Solution Here is the proof $B_1, B_2,\ B_3,\ B_4$

B_1: $(\mathbf{L}a \cup \neg \mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg \mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

Part 2

Verify whether the system S is **M**-sound.

You can use **shorthand notation**

Solution

Observe that both logical axioms of S are **M tautologies**

A1 is **M** tautology by definition of the semantics, **A1** is **M** tautology by direct eveluation

Rule r1 is **sound** because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$

Rule 2 is **not sound** because when $A = T$ and $B = F$ (or $B = \perp$) we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$ or $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

PROBLEM 5 (Extra Credit)

Part 1

If the system S **is not sound/ sound** under your semantics **M** then **re-define the connectives** in a way that such obtained new semantics **N** would make S S **sound/not sound**

You can use **shorthand notation**

Solution To make rule $r2$ sound while preserving the "soundness off axioms we have to modify ONLY the definition of implication. Here is the **N** semantics implication

N- Implication

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	T	$\perp$	T
T	T	T	T

Remark that it would be hard to convince anybody to use our sound proof system it as it would be hard to convince anybody to adopt our **N** semantics!

Part 2

Give an **example** on an infinite, **M**-consistent set of formulas of the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$

Solution

Take **G** be a set of all propositional variables, i.e. **G** = VAR

v such that v(a) = T for all $a \in VAR$ is obviously a **M** model for **G** and it proves that **G** is **M**-consistent