PART 1: DEFINITIONS  As in Lectures

PART 2: PROBLEMS

PROBLEM 1

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following:
one does not like to play bridge or one likes not to play bridge
as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$, where $\mathbf{L} \ A$ represents statement ”one likes $A$”, ”$A$ is liked”.

Solution  We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b$

$a$ denotes statement: play bridge,
$b$ denotes a statement: the weather is good

Translation 1

$A_1 = (\mathbf{L}a \lor (b \Rightarrow (\neg Ia \lor \mathbf{L} \neg a)))$

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$.

Solution  We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$ as follows.

Propositional Variables: $a, b, c$

$a$ denotes statement: One likes to play bridge ,
$b$ denotes a statement: the weather is good , and
$c$ denotes a statement: one likes not to play bridge

Translation 2:

$A_2 = (a \lor (b \Rightarrow (\neg a \lor c)))$

PROBLEM 2

CREATE YOUR OWN 3 valued extensional semantics $M$ for the language $\mathcal{L}_{\{\neg, \land, \lor, \Rightarrow\}}$ by defining the connectives $\neg, \land, \lor, \Rightarrow$ on a set $\{F, \bot, T\}$ of logical values.

You must follow the following assumptions

Assumption 1

The third logical value value is intermediate between truth and falsity, i.e. the set of logical values is ordered and we have the following $F < \bot < T$

Assumption 2  $T$ is the designated value
2. The semantics has to **model the situation** in which one "likes" only truth; i.e. in which 
\[ LT = T \quad \text{and} \quad L \perp = F, \quad LF = F \]

3. The **connectives** \( \neg, \cup, \Rightarrow \) can be defined as you wish, but you have to define them in such a way to make sure that 
\[ \models_M (LA \cup \neg LA) \]

**Part 1** Write down definition of logical connectives

**Solution**

Here is MY \( M \) semantics - yours can be different!

I define the logical connectives by "shorthand" writing functions defining connectives in form of the "truth tables" and skipping other points of the definition - as I have typed it so many times for you before!

<table>
<thead>
<tr>
<th>L Connective</th>
<th>Negation :</th>
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<tbody>
<tr>
<td>L</td>
<td>(~)</td>
</tr>
<tr>
<td>F</td>
<td>F \quad F \quad T</td>
</tr>
<tr>
<td>F</td>
<td>T \quad F \quad T</td>
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</table>

<table>
<thead>
<tr>
<th>Implication</th>
<th>Disjunction :</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Rightarrow )</td>
<td>( \cup )</td>
</tr>
<tr>
<td>F</td>
<td>F \quad F \quad T</td>
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<tr>
<td>T</td>
<td>T \quad T \quad T</td>
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</tbody>
</table>

**Part 2**

Verify whether \[ \models_M (LA \cup \neg LA) \] under your semantics - you can use shorthand notation

**Solution**

We verify
\[ LT \cup \neg LT = T \cup F = T, \quad L \perp \cup \neg L \perp = F \cup \neg F = F \cup T = T, \quad LF \cup \neg LF = F \cup \neg F = T \]

**PROBLEM 3**

**Part 1** Verify whether the formulas \( A_1 = (La \cup (b \Rightarrow (\neg La \cup L \neg a))) \) and \( A_2 = (a \cup (b \Rightarrow (\neg a \cup c))) \) have a model/counter model under your semantics \( M \). You can use shorthand notation.

**Solution**

Given \( A_1 = (La \cup (b \Rightarrow (\neg La \cup L \neg a))) \), and 
\[ A_2 = (a \cup (b \Rightarrow (\neg a \cup c))) \]

Any \( v \), such that \( v(a) = T \) is a \( M \) **model** for \( A_1 \) and for \( A_2 \) directly from the definition of \( \cup \).
Part 2  Verify whether the following set $G$ is $M$-consistent. You can use shorthand notation
\[ G = \{ \text{La, } (a \cup \neg Lb), \ (a \Rightarrow b), \ b \} \]

Solution
Any $v$, such that $v(a) = T$, $v(b) = T$ is a $M$ model for $G$ as
\[ LT = T, \ (T \cup \neg LT) = T, \ (T \Rightarrow T) = T, \ b = T \]

PROBLEM 4
Let $S$ be the following proof system
\[ S = ( \mathcal{L}_{\setminus \neg, \lor, \Rightarrow}, \ F, \ \{A1, A2\}, \ \{r1, r2\} ) \]
for the logical axioms and rules of inference defined for any formulas $A, B \in F$ as follows

Logical Axioms
A1  $(\text{La} \cup \neg \text{La})$
A2  $(A \Rightarrow \text{La})$

Rules of inference:
\[ \begin{align*}
(r1) & \quad A ; B \\
     & \quad (A \cup B) \\
(r2) & \quad A \\
     & \quad \text{L}(A \Rightarrow B)
\end{align*} \]

Part 1
Show, by constructing a proper formal proof that
\[ \vdash_S ((\text{Lb} \cup \neg \text{Lb}) \cup \text{L}((\text{La} \cup \neg \text{La}) \Rightarrow b)) \]
You must write comments how each step pot the proof was obtained
Write all steps of the formal proof as follows- write as MANY as you NEED!

Solution Here is the proof $B_1, B_2, B_3, B_4$
\[ \begin{align*}
B_1: & \quad (\text{La} \cup \neg \text{La}) \quad \text{Axiom } A_1 \text{ for A= } a \\
B_2: & \quad \text{L}((\text{La} \cup \neg \text{La}) \Rightarrow b) \quad \text{rule r2 for B=b applied to } B_1 \\
B_3: & \quad (\text{Lb} \cup \neg \text{LAb}) \quad \text{Axiom } A_1 \text{ for A=b} \\
B_4: & \quad ((\text{Lb} \cup \neg \text{Lb}) \cup \text{L}((\text{La} \cup \neg \text{La}) \Rightarrow b)) \quad \text{r1 applied to } B_3 \text{ and } B_2
\end{align*} \]

Part 2
Verify whether the system $S$ is $M$-sound.
You can use shorthand notation

Solution
Observe that both logical axioms of $S$ are $M$ tautologies
**PROBLEM 5** (Extra Credit)

**Part 1**

If the system $S$ is not sound/ sound under your semantics $M$ then re-define the connectives in a way that such obtained new semantics $N$ would make $S$ sound/not sound.

You can use shorthand notation.

**Solution** To make rule $r_2$ sound while preserving the "soundness off axioms we have to modify ONLY the definition of implication. Here is the $N$ semantics implication.

**N- Implication**

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<thead>
<tr>
<th>⇒</th>
<th>F</th>
<th>⊥</th>
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<tr>
<td>F</td>
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**Remark** that it would be hard to convince anybody to use our sound proof system it as it would be hard to convince anybody to adopt our $N$ semantics!

**Part 2**

Give an example on an infinite, $M$-consistent set of formulas of the language $L_{\{\neg, \vee, \cup, \Rightarrow\}}$.

**Solution**

Take $G$ be a set of all propositional variables, i.e. $G = VAR$.

$v$ such that $v(a) = T$ for all $a \in VAR$ is obviously a $M$ model for $G$ and it proves that $G$ is $M$-consistent.