CSE371 Short Final Review - covers material from Q5, Q6 Fall 2015

QUESTION 1

H is the following proof system:

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad A1, A2, A3,\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP Rule of inference:

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

We know that S is SOUND and COMPLETE under classical semantics.

Show whether S is sound/not sound under **M** semantics defined below.

M Negation: $\neg F = T,\ \neg \perp = \perp, \neg T = F$,

M Conjunction: for any $a, b \in \{F, \perp, T\}$, $a \cap b = min\{a, b\}$,

M Disjunction: for any $a, b \in \{F, \perp, T\}$, $a \cup b = max\{a, b\}$,

M Implication: for any $a, b \in \{F, \perp, T\}$, $a \Rightarrow b = \neg a \cup b$.

Solution S is NOT SOUND. Axiom $A1$ is not a M-tautology. If $A = \perp$ and $B = \perp$, then $(\perp \Rightarrow (\perp \Rightarrow \perp)) = \perp$.

QUESTION 2

Remark This problem is taken straight from the BOOK and your exercises solutions! I write the solution to spare your time!

Let $S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ \ \mathbf{A1}, \mathbf{A2}, \mathbf{A3},\ MP)$ be a proof system with the following axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The following Lemma holds in the system S.

LEMMA

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

Solution

B_1 $(A \Rightarrow B)$
Hypothesis

B_2 $(\neg\neg A \Rightarrow A)$

Already PROVED

B_3 $(\neg\neg A \Rightarrow B)$

Lemma **a** for $A = \neg\neg A, B = A, C = B$

B_4 $(B \Rightarrow \neg\neg B)$

Already PROVED

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

Lemma **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Already PROVED

Example 4 ch8 for $B = \neg A, A = \neg B$

B_7 $(\neg B \Rightarrow \neg A)$

B_5, B_6 and MP

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem

QUESTION 3

Consider a system **RS3** obtained from **RS** by changing the sequence $\Gamma^{'}$ into Γ in all of the rules of inference of **RS**.

1. Define SHORTLY Decomposition Tree for any A in **RS3**

Solution

The decomposition tree is a slight modification of definition of **RS** tree; now we can decompose any decomposable formula at the decomposable node. Peaae write down carefully your definition.

2. Show an example of a formula and its 2 decomposition trees

Solution

You can use **any formula** that leads to a node with at least two decomposable formulas.

item[3.] Prove Completeness Theorem for **RS3**. We assume that the STRONG soundness has been proved.

Solution

The proof is a gain a modification of **RS** proof.

Given **any formula** $A \in \mathcal{F}$

Assume $\nvdash_{RS3} A$, i.e. A **does not have a proof** in RS3

Let $\mathcal{T}_A$ be a set of **all decomposition trees** of A. As $\nvdash_{RS3} A$, **each** $T \in \mathcal{T}_A$ has a non-axiom leaf.

We choose an arbitrary $T_A \in \mathcal{T}_A$.

The non-axiom leaf L_A **defines** a truth assignment v which falsifies A, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \textbf{anyvalue} & \text{if a does not appear in } L_A \end{cases}$$

QUESTION 4 Consider a system **RS1** obtained from **RS** by changing the sequence Γ' into Γ and Δ into Δ' in all of the rules of inference of **RS**.

2. Define in your own words, for any A, the decomposition tree $\mathbf{T}_A$ in **RS1**.

Solution FACT 1: Steps are as follows.

1.Decompose using rule defined by the main connective of A

2. Scan resulting sequence from RIGHT to LEFT and find first decomposable formula A

3. Repeat 1. and 2. until no more decomposable formulas.

End of Tree Construction

FACT 1: Tree $\mathbf{T}_A$ is a proof if all leaves are axioms.

FACT 2: The proof does not exist otherwise, i.e. when there is a non- axiom leaf because the tree because as in **RS**, the decomposition tree is unique.

Observe that we need **Facts 1, 2** in order to prove Completeness Theorem.

QUESTION 5

Let **GL** be the Gentzen style proof system for classical logic.

(1) Prove, by constructing a proper decomposition tree that

$\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$.

Solution Consider the following tree.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b)))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg b \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)$$

$$\neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a, \neg b$$

$$\mid (\rightarrow \neg)$$

$$b, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$b, a, \neg b, (\neg(a \cap b) \Rightarrow b) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$b, a, (\neg(a \cap b) \Rightarrow b) \longrightarrow b$$

$$\bigwedge (\Rightarrow \rightarrow$$

$$b, a \longrightarrow \neg(a \cap b), b \qquad\qquad b, a, b \longrightarrow b$$

$$\mid (\rightarrow \neg) \qquad\qquad axiom$$

$$b, a, (a \cap b) \longrightarrow b$$

$$\mid (\cap \rightarrow)$$

$$b, a, a, b \longrightarrow b$$

$$axiom$$

All leaves of the decomposition tree are axioms, hence the proof has been found.

(2) Use the **completeness theorem** for **GL** to prove that

$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

Solution

By the **Completeness Theorem** we have that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \quad \textit{if and only if} \quad \not\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Any v, such that $v(a) = v(b) = F$ is a counter-model for $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$, hence By the **Completeness Theorem** $\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

QUESTION 6

Let **GL** be the Gentzen style proof system for classical logic.

1. Define SHORTLY Decomposition Tree for any A in **GL**.

Solution

Here is my short definition.

Decomposition Tree $\mathbf{T}_A$

For each formula $A \in \mathcal{F}$, a decomposition tree $\mathbf{T}_A$ is a tree build as follows.

Step 1. The sequent $\longrightarrow A$ is the **root** of $\mathbf{T}_A$ and for any node $\Gamma \longrightarrow \Delta$ of the tree we follow the steps below.

Step 2. If $\Gamma \longrightarrow \Delta$ is indecomposable, then $\Gamma \longrightarrow \Delta$ becomes a **leaf** of the tree.

Step 3. If $\Gamma \longrightarrow \Delta$ is decomposable, then we pick one rule that applies by matching the sequent of the current node with the domain of the rules. Then we apply this rule as decomposition rule and put its left and right premises as the left and right leaves, or as one leaf in case of one premiss rule.

Step 4. We repeat steps 2 and 3 until we obtain only indecomposable leaves.

2. Prove Completeness Theorem for **GL**. We assume that the STRONG soundness has been proved.

Solution

Formula Completeness Theorem

For any formula $A \in \mathcal{F}$,

$$\vdash_{\mathbf{GL}} A \;\; iif \quad and \quad only \;\; if \quad \models A.$$

We prove the logically equivalent form of the Completeness part: for any $A \in \mathcal{F}$

$$\text{If} \quad \nvdash_{\mathbf{GL}} \longrightarrow A \;\; \text{then} \;\; \not\models A.$$

Assume $\nvdash_{\mathbf{GL}} \longrightarrow A$, i.e. $\longrightarrow A$ **does not have a proof** in **GL**. Let $\mathcal{T}_A$ be a set of **all decomposition trees** of $\longrightarrow A$. As $\nvdash_{\mathbf{GL}} \longrightarrow A$, each $\mathcal{T} \in \mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $T_A \in \mathcal{T}_A$. Let $\Gamma^{'} \longrightarrow \Delta^{'}$, $\Gamma^{'}$ be an non-axiom leaf of T_A, for $\Delta^{'} \in VAR^*$ such that $\{\Gamma^{'}\} \cap \{\Delta^{'}\} = \emptyset$.

The non-axiom leaf $L = \Gamma^{'} \longrightarrow \Delta^{'}$ *defines* a truth assignment $v : VAR \leftarrow \{T, F\}$ which falsifies A as follows:

$$v(a) = \begin{cases} T & \text{if a appears in } \Gamma^{'} \\ F & \text{if } a \text{ appears in} \Delta^{'} \\ any\ value & \text{if a does not appear in } L \end{cases}$$

This proves, by **strong soundness** of the rules of inference of **GL** that $\not\models A$.

QUESTION 7

PROVE without USE of the **Completeness Theorem** that $\nvdash_{\mathbf{GL}}((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

Solution Consider the following trees.

$\mathbf{T}_1$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (a \Rightarrow b) \longrightarrow a$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge(\Rightarrow\rightarrow$$

$$\longrightarrow a, b, a \qquad\qquad b \longrightarrow b, a$$

$$non - axiom \qquad\qquad axiom$$

The tree has a non- axiom leaf, so it is not a proof. To prove that a proof does not exist in GL we must consider all possible decomposition trees; in this case there is only one more (the second choice in the second step).

$$\mathbf{T}_1$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\rightarrow$$

$$\longrightarrow (\neg b \Rightarrow a), a \qquad\qquad b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow) \qquad\qquad \mid (\rightarrow\Rightarrow)$$

$$\neg b \longrightarrow a, a \qquad\qquad b, \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow) \qquad\qquad \mid (\neg \rightarrow)$$

$$\longrightarrow b, a, a \qquad\qquad b \longrightarrow b, a$$

$$non - axiom \qquad\qquad axiom$$

These are **all possible** decomposition trees and none is a proof; hence the **proof does not exist**.