CSE371 Q5 Fall 2015
(20pts + 5 extra pts)

NAME **ID:** **Math/CS**

QUESTION 2

H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3, A4\},\ \ MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

Solution: Soundness Theorem holds because all axioms of H are tautologies and MP leads from tautologies to a tautology.

(2) Why Deduction Theorem holds for H?

Solution: System H extends by one extra axiom $A3$ the proof system H_1 for which we have proved the deduction theorem.

(3) Is H COMPLETE?

Solution: YES. Axioms A1, A2, A3 of H are axioms of the system H_2 from Chapter 8. It is stated in Chapter 8 and proved in Chapter 9 that Completeness Theorem holds for H_2.

4. Prove the following: $A \vdash_H (A \Rightarrow A)$

Solution 1: Proof is as follows.

$B_1 =\ (A \Rightarrow (A \Rightarrow A))$
Axiom $A1$ for $B = A$

$B_2 =\ A$
Hypothesis

$B_3 =\ (A \Rightarrow A)$
B_1, B_2 and MP

Solution 2: We use Deduction Theorem.

$A \vdash_H (A \Rightarrow A)$ if and only if $\vdash_H (A \Rightarrow (A \Rightarrow A))$, what is true because $(A \Rightarrow (A \Rightarrow A))$ is axiom $A1$. The proof is one element sequence:

$B_1 = A \Rightarrow (A \Rightarrow A)$
Axiom $A1$ for $B = A$

QUESTION 2

Complete the steps $B_1, ..., B_5$ of the formal proof in H_2 of $(B \Rightarrow \neg\neg B)$ by writing all details for each step of the proof.

You can use the following already proved facts:

F1 $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

F2 $\vdash_{H_2} (\neg\neg B \Rightarrow B)$

Here are the steps

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

$B_5 = (B \Rightarrow \neg\neg B)$

Solution The completed comments are as follows.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
Already proved fact: $\vdash_{H_2} (\neg\neg B \Rightarrow B)$ for $B = \neg B$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B);\ ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$
Axiom $A1$ for $A = B, B = \neg\neg\neg B$

$B_5 = \ (B \Rightarrow \neg\neg B)$

B_3, B_4 and already proved fact:
$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$ for
$A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$ i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$

QUESTION 3

Use Deduction Theorem to prove

F1 $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

Solution

$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$ iff $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_2} C$

B_1 $(A \Rightarrow B)$ Hyp

B_2 A Hyp

B_3 B MP on B_2, B_1 item[] B_4 $(B \Rightarrow C)$ Hyp

B_5 C MP on B_3, B_4