PART 1: DEFINITIONS  20pts

D1  5pts
Given a language $L\{\Rightarrow, \cup, \cap, \neg\}$ and a formula $A$ of this language
Write a definition of $\models A$

D2  5pts
Given two languages: $L_1 = L_{CON_1}$ and $L_2 = L_{CON_2}$, for $CON_1 \neq CON_2$
Write definition of logical equivalence of $L_1$ and $L_2$, i.e. the definition of $L_1 \equiv L_2$
Given a proof system \( S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R}) \) and any set \( \Gamma \subseteq \mathcal{E} \) and any expression \( E \in \mathcal{E} \)
Write the definition of \( \Gamma \vdash_S E \)

Write the statements of **Soundness Theorem** and of **Completeness Theorem** for a proof system \( S \) and semantics \( M \)

**Soundness Theorem**

**Completeness Theorem**
PART 2: PROBLEMS

QUESTION 1  20pts

1. Find a classical restricted model and restricted counter-model for $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

2. Given a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

Show that the set $\mathcal{G}$ is consistent under classical semantics.
We define H semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = \max\{a, b\}, \quad a \cap b = \min\{a, b\}.$$  

The Truth Tables for Implication and Negation are:

**H-Implication**

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<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\bot$</th>
<th>T</th>
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<td>F</td>
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<td>$\bot$</td>
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<td>T</td>
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**H Negation**

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
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<td>T</td>
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**QUESTION 2**

We know that $v : VAR \rightarrow \{ F, \bot, T \}$ is such that

$v^*( (a \cap b) \Rightarrow (a \Rightarrow c)) = \bot$ under H semantics.

Use this information to evaluate $v^* ( (b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

You can use shorthand notation.
QUESTION 3  20pts

1. Write the formula \((a \Rightarrow (b \cap \neg b)) \Rightarrow a\) as a formula of the language \(L_{\{\neg, \cup\}}\), i.e. as a formula \(B\) of a language \(L_{\{\neg, \cup\}}\), such that \(A \equiv B\).

Write down all logical equivalences you need while solving this problem.

2. Prove that \(L_{\{\neg, \cup\}} \equiv L_{\{\neg, \cap\}}\)
QUESTION 4  20pts

$S$ is the following proof system:

$$S = (L_{\rightarrow, \lor, \neg}, \mathcal{F}, LA = \{(A \Rightarrow (A \cup B))\} \ (r1), \ (r2)$$

Rules of inference:

$$\frac{A : B}{(A \lor \neg B)} \quad \frac{A : (A \lor B)}{B}$$

1. Verify whether $S$ is sound/not sound under classical semantics.

2. Find a formal proof of $\neg(A \Rightarrow (A \cup B))$ in $S$, i. e. show that $\vdash_S \neg(A \Rightarrow (A \cup B))$

3. Explain whether the above point 2. proves that $\models \neg(A \Rightarrow (A \cup B))$?
EXTRA CREDIT  25pts

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and the use of some basic tautologies.

Consider any proof system $S$,

$$S = (\mathcal{L}_{\land, \lor, \Rightarrow, \neg}, F, LA, (MP) \frac{A, (A \Rightarrow B)}{B})$$

that is complete under classical semantics

Definition 1 Let $X \subseteq F$ be any subset of the set $F$ of formulas of the language $\mathcal{L}_{\land, \lor, \Rightarrow, \neg}$ of $S$

We define a set $Cn(X)$ of all consequences of the set $X$ as follows

$$Cn(X) = \{A \in F : X \vdash_S A\}$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in $S$ from the set $(LA \cup X)$.

Part 1  5pts

1. Prove that for any subsets $X, Y$ of the set $F$ of formulas the following monotonicity property holds.
   If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

2.  5pts
   Prove that for any $X \subseteq F$, the set $T$ of all propositional classical tautologies is a subset of $Cn(X)$, i.e.
   $$T \subseteq Cn(X)$$
3. 15pts

Prove that for any $A, B \in F$, $X \subseteq F$,

$$(A \cap B) \in Cn(X) \text{ if and only if } A \in Cn(X) \text{ and } B \in Cn(X)$$

**Hint:** Use the **Monotonicity** and **Completeness** of $S$ i.e. the fact that any tautology you might need is provable in $S$. 