Problem 1a

Given a formula \( A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) \)

and its restricted model \( v_A : \{a, b, c\} \rightarrow \{T, F\} \), \( v_A(a) = T, v_A(b) = T, v_A(c) = F \)

Extend \( v_A \) to the set of all propositional variables VAR to obtain 2 different, non restricted models for A

Solution

Model \( w_1 \) is a function \( w_1 : VAR \rightarrow \{T, F\} \) such that

\[
\begin{align*}
w_1(a) &= v_A(a) = T, \\
w_1(b) &= v_A(b) = T, \\
w_1(c) &= v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \ x \in VAR - \{a, b, c\}
\end{align*}
\]

Model \( w_2 \) is defined by a formula

\[
\begin{align*}
w_2(a) &= v_A(a) = T, \\
w_2(b) &= v_A(b) = T, \\
w_2(c) &= v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \ x \in VAR - \{a, b, c\}
\end{align*}
\]

Problem 1b

1. Give an example of an infinite set of formulas of \( L_{\{\neg, \cup\}} \), different from the set \( T \) of its tautologies that has a model under classical semantics

Reminder: a set \( G \subseteq F \) of formulas is called consistent if and only if \( G \) has a model

Solution

There plenty of examples; here is the simplest one: \( G = \forall AR \)

\[ v : VAR \rightarrow \{T, F\}, \quad \text{such that} \quad v(x) = T \quad \text{for all} \ x \in VAR \] is obviously a model for each formula in \( G \) and hence by definition is a model for \( G \)

2. Give an example of an infinite set of formulas of \( L_{\{\neg, \cup\}} \), different from the set \( C \) of its contradictions that is inconsistent

Reminder: a set \( G \subseteq F \) is called inconsistent if and only if \( G \) does not have a model

Solution

There plenty of examples; here is the simplest one:

Let \( c \) be any variable, i.e. \( c \in VAR \), we take

\[ G = VAR \cup \{c, \neg c\} \]
Obviously, the finite set \( \{c, \neg c\} \) does not have a model, and hence the infinite set \( VAR \cup \{c, \neg c\} \) does not have a model and hence, by definition is inconsistent

Problem 2

Definition Let \( S_3 \) be a 3-valued semantics for \( L_{\neg, \cup, \Rightarrow} \) defined as follows.

\[
a \Rightarrow b = \neg a \cup b, \quad \text{for any } a, b \in \{F, U, T\} \quad \text{where } F < U < T, \text{ and}
\]

\[
\begin{array}{c|ccc}
\cup & F & U & T \\
F & F & U & T \\
U & U & U & U \\
T & T & U & T \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\neg & F & U & T \\
T & F & U & U \\
\end{array}
\]

Consider the following classical tautologies: \( A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \)

(a) Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist.

You can use shorthand notation.

Solution Any \( v \) such that \( v(a) = v(b) = U \) is a counter-model for both \( A_1 \) and \( A_2 \), as

\[
v^*(a \cup \neg a) = U \cup \neg U = U \cup T = U \not= T
\]

\[
v^*(a \Rightarrow (b \Rightarrow a)) = (U \Rightarrow (U \Rightarrow U)) = U \Rightarrow U = \neg U \cup U = F \cup U \not= T
\]

(b) Define your own 2-valued semantics \( S_2 \) for \( L \), such that none of \( A_1, A_2 \) is a \( S_2 \) tautology

Verify your results. You can use shorthand notation.

Solution This is not the only solution, but it is the simplest and most obvious! Here it is.

We define \( \neg a = F, \quad a \Rightarrow b = F \quad a \cup b = F \) for all \( a, b \in \{F, T\} \)

Obviously, for any \( v \),

\[
v^*(a \cup \neg a) = F \quad \text{and} \quad v^*(a \Rightarrow (b \Rightarrow a)) = F
\]