Chapter 12: Gentzen Sequent Calculus for Intuitionistic Logic

Part 1: LI System

The proof system LI was published by Gentzen in 1935 as a particular case of his proof system LK for the classical logic.

We discussed a version of the original Gentzen’s system LK in the previous chapter.

We present now the proof system LI and then we show how it can be extended to the original Gentzen system LK.
Language of LI

We consider the set of all Gentzen sequents built out of the formulas of our language $\mathcal{L}$ and the additional symbol $\rightarrow$, as defined in the previous section:

$$SEQ = \{ \Gamma \rightarrow \Delta : \Gamma, \Delta \in \mathcal{F}^* \}.$$

In the intuitionistic logic we deal only with sequents of the form

$$\Gamma \rightarrow \Delta,$$

where $\Delta$ consists of at most one formula.

The intuitionistic sequents are elements of a following subset $IS$ of the set $SEQ$ of all sequents.

$$ISEQ = \{ \Gamma \rightarrow \Delta : \Delta \text{ consists of at most one formula} \}.$$
Axioms of LI consists of any sequent from the set $ISEQ$ which contains a formula that appears on both sides of the sequent arrow $\rightarrow$, i.e any sequent of the form

$$\Gamma_1, A, \Gamma_2 \rightarrow A.$$ 

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules.
Structural Rules of LI

Weakening

\[
(\rightarrow\text{weakening}) \quad \frac{\Gamma \rightarrow A}{\Gamma \rightarrow \Gamma \rightarrow A}
\]

\(A\) is called the weakening formula.
Contraction

\[(\text{contr} \rightarrow) \quad \frac{A, A, \Gamma \quad \rightarrow \quad \Delta}{A, \Gamma \quad \rightarrow \quad \Delta}\]

\(A\) is called the contraction formula.

\(\Delta\) contains at most one formula.

Exchange

\[(\text{exchange} \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \quad \rightarrow \quad \Delta}{\Gamma_1, B, A, \Gamma_2 \quad \rightarrow \quad \Delta',}\]

\(\Delta\) contains at most one formula.
Logical Rules of LI

Conjunction rules

\[ (\cap \rightarrow) \quad \frac{A, B, \Gamma \rightarrow \Delta}{(A \cap B), \Gamma \rightarrow \Delta} , \]

\[ (\rightarrow \cap) \quad \frac{\Gamma \rightarrow A ; \Gamma \rightarrow B}{\Gamma \rightarrow (A \cap B)} , \]

Disjunction rules

\[ (\rightarrow \cup)_1 \quad \frac{\Gamma \rightarrow A}{\Gamma \rightarrow (A \cup B)} , \]

\[ (\rightarrow \cup)_1 \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow (A \cup B)} , \]

\[ (\cup \rightarrow) \quad \frac{A, \Gamma \rightarrow \Delta ; B, \Gamma \rightarrow \Delta}{(A \cup B), \Gamma \rightarrow \Delta} , \]

\(\Delta\) contains at most one formula.
Implication rules

\[
\Gamma \rightarrow (A \Rightarrow B),
\]

\[
\Gamma \rightarrow A; B, \Gamma \rightarrow \Delta
\]

\[
(A \Rightarrow B), \Gamma \rightarrow \Delta
\]

\(\Delta\) contains at most one formula.

Negation rules

\[
\Gamma \rightarrow A
\]

\[
\neg A, \Gamma \rightarrow
\]

\[
A, \Gamma \rightarrow \neg A.
\]

We define

\[
\text{LI} = (\mathcal{L}, ISEQ, AL, \text{Structural rules, Logical rules } \}
\]
LK - Original Gentzen system for the classical propositional logic.

Language of LK: $\mathcal{L} = \mathcal{L}\{\neg, \cap, \cup, \Rightarrow\}$, and $\mathcal{E} = \text{SEQ}$, for

$$\text{SEQ} = \{ \Gamma \rightarrow \Delta : \Gamma, \Delta \in \mathcal{F}^* \}.$$ 

Axioms of LK: any sequent of the form

$$\Gamma_1, A, \Gamma_2 \rightarrow \Gamma_3, A, \Gamma_4.$$
Rules of inference of LK

1. We adopt all rules of LI with no intuitionistic restriction that the sequence $\Delta$ in the succedent of the sequence is at most one formula.

2. We add two structural rules to the system LI.

We add one more contraction rule:

$$\frac{\Gamma \rightarrow \Delta, A, A,}{\Gamma \rightarrow \Delta, A},$$

We add one more exchange rule:

$$\frac{\Delta \rightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \rightarrow \Gamma_1, B, A, \Gamma_2}.$$
Observe that they both become obsolete in LI.

The rules of inference of LK are hence as follows.

Structural Rules of LK

Weakening

\[
\begin{align*}
\text{(weakening } \rightarrow \text{)} & \quad \frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}, \\
\text{(→ weakening)} & \quad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}.
\end{align*}
\]

A is called the weakening formula.
Contraction

\[
(\text{contr} \to) \quad \frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta},
\]

\[
(\to \text{contr}) \quad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A},
\]

\[A\text{ is called} \text{ the contraction formula.}\]

Exchange

\[
(\text{exchange} \to) \quad \frac{\Gamma_1, A, B, \Gamma_2 \to \Delta}{\Gamma_1, B, A, \Gamma_2 \to \Delta},
\]

\[
(\to \text{exchange}) \quad \frac{\Delta \to \Gamma_1, A, B, \Gamma_2}{\Delta \to \Gamma_1, B, A, \Gamma_2}.
\]
Logical Rules of LK

Conjunction rules

\[(\cap \rightarrow) \quad \frac{A, B, \Gamma \rightarrow \Delta}{(A \cap B), \Gamma \rightarrow \Delta},\]

\[(\rightarrow \cap) \quad \frac{\Gamma \rightarrow \Delta, A ; \Gamma \rightarrow \Delta, B, \Delta}{\Gamma \rightarrow \Delta, (A \cap B)}\]

Disjunction rules

\[(\rightarrow \cup) \quad \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, (A \cup B)},\]

\[(\cup \rightarrow) \quad \frac{A, \Gamma \rightarrow \Delta ; B, \Gamma \rightarrow \Delta}{(A \cup B), \Gamma \rightarrow \Delta}.\]
Implication rules

\[
\frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, (A \Rightarrow B)},
\]

\[
\frac{\Gamma \rightarrow \Delta, A ; B, \Gamma \rightarrow \Delta}{(A \Rightarrow B), \Gamma \rightarrow \Delta}.
\]

Negation rules

\[
\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta},
\]

\[
\frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}.
\]

We define formally \( LK = (\mathcal{L}, SEQ, AL, \text{ Structural rules, Logical rules}) \).