

# Propositional Resolution

## Part 1

Short Review

Professor Anita Wasilewska  
CSE 352 Artificial Intelligence

# SYNTAX “dictionary”

**Literal** – any propositional VARIABLE  $a$  or negation of a variable  $\neg a$ ,  $a \in \text{VAR}$ ,

**Example** - variables:  $a, b, c$ , negation of variables:  $\neg a, \neg b, \neg c \dots$

**Positive Literal** – any variable  $a \in \text{VAR}$

**Clause** – any **finite set** of **literals**

**Example-**  $C_1, C_2, C_3$  are clauses where

$$C_1 = \{a, b\}, C_2 = \{a, \neg c\}, C_3 = \{a, \neg a, \dots, a_k\}$$

# Syntax “Dictionary”

**Empty Clause**  $\{\}$  – is an empty set i.e. a clause without elements.

**Finite set of clauses**

$$CL = \{ C1, \dots, Cn \}$$

**Example**

$$CL = \{ \{a\}, \{ \}, \{ b, \neg a \}, \{ c, \neg d \} \}$$

# Semantics – Interpretation of Clauses

- Think semantically of a clause
  - $C = \{ a_1, \dots, a_n \}$  as disjunction, i.e.  
C is logically equivalent to  
 $a_1 \cup a_2 \cup \dots \cup a_n$   $a_i \in \text{Literal}$
  - Formally – a truth assignment  $v : \text{VAR} \rightarrow \{0, 1\}$  we extended it to set of all CLAUSES **CL** as follows:  
 $v^* : \text{CL} \rightarrow \{0, 1\}$  we extend  
 $v^*(C) = v^*(a_1) \cup \dots \cup v^*(a_n)$
- for any clause C in **CL**, where  
0 – False, 1 – True
- Shorthand :  $v^* = v$

# Satisfiability, Model, Tautology

**Example:** let  $v : \text{VAR} \rightarrow \{0, 1\}$  be such that

- $v(a) = 1, v(b) = 1, v(c) = 0$

$$C = \{ a, \neg b, c, \neg a \}$$

We evaluate :

$$v(C) = v(a) \cup \neg v(b) \cup v(c) \cup \neg v(a) =$$

$$1 \cup 0 \cup 0 \cup 1 = 1$$

**OBSERVE** that  $v(C) = 1$  for all  $v$ , i.e.

$C = \{ a, \neg b, c, \neg a \}$  is a **Tautology**

# Satisfiability, Model, Tautology

For any clause  $C$ , and any truth assignment  $v$  we write  $v \models C$  and say that  $v$  satisfies  $C$  iff  $v(C) = 1$

Any  $v$  such that  $v \models C$  is called a **MODEL** for  $C$

A clause  $C$  is **satisfiable** iff it has a **MODEL**, i.e.

$C$  is **satisfiable** iff there is a  $v$  such that  $v \models C$

A clause  $C$  is a **tautology** iff  $v \models C$  for all  $v$ ,  
i.e all truth assignments  $v$  are **models for  $C$**

# Notations

- $a, a, a$  (finite sequence of 3 elements)
- $\{a, a, a\} = \{a\}$  finite set
- $a, b, c \neq b, a, c$  (different sequences)
- $\{a, b, c\} = \{b, a, c\}$  (same sets)
- $\{a, a, b, c\}$  (multi - sets)

# Sets of Clauses CL

A clause **C** is **unsatisfiable** iff it has **no MODEL**  
i.e.  $v(c) = 0$  for all truth assignments **v**

**Remark:** the empty clause **{}** is the only **unsatisfiable** clause

Let **CL** =  $\{ C_1, \dots, C_n \}$  be a **finite set of clauses**.

We extended  $v : \text{VAR} \rightarrow \{0, 1\}$  to any set of clauses **CL**

$$v(\mathbf{CL}) = v(c_1) \wedge \dots \wedge v(c_n)$$

A finite set of clauses **CL** is semantically equivalent to a conjunction of all clauses in the set **CL**.



# Unsatisfiability

A set of clauses **CL** is **satisfiable**

iff it **has a model**, i.e. iff  $\exists v \ v(\mathbf{CL}) = 1$ .

A set of clauses **CL** is **unsatisfiable**

iff it **does not have a model**, i.e. iff  $\forall v \ v(\mathbf{CL}) = 0$ .

Remark:

If  $\{\} \in \mathbf{CL}$  then **CL** is unsatisfiable.

# Unsatisfiability

Consider a set of clauses

$$\mathbf{CL} = \{\{a\}, \{a, b\}, \{\neg b\}\}$$

$\mathbf{CL}$  is **satisfiable** because  $v$ , such that  $v(a) = 1$ ,  
 $v(b) = 0$  is a **model** for  $\mathbf{CL}$

Check:  $v(\mathbf{CL}) = 1 \wedge (1 \cup 0) \wedge 1 = 1$

**Remark:** When  $\{a\}$  and  $\{\neg a\}$  are in it, then  
the set  $\mathbf{CL}$  is **unsatisfiable**

Remember:  $(a \wedge \neg a)$  is a contradiction

# Syntax and Semantics

- Example:
- $C1 = \{ a, b, \neg c \}, C2 = \{ c, a \}$  - **syntax**
- $C1 = a \cup b \cup \neg c$  - **semantics**
- $C2 = c \cup a$  - **semantics**
  
- $CL = \{C1, C2\} = \{ \{a, b, \neg c\}, \{c, a\} \}$  - **syntax**
  
- $CL = (a \cup b \cup \neg c) \wedge (c \cup a)$  - **semantics**

# Syntax and Semantics

## Definitions:

CL is **satisfiable** iff there is  $v$ , such that  $v( CL ) = 1$

CL is **unsatisfiable** iff for all  $v$ ,  $v( CL ) = 0$

- $CL = \{ C1, C2, \dots, Cn \}$  - **synatx**
- $CL = C1 \wedge \dots \wedge Cn$  - **semantics**

# Semantical Decidability

- A statement:
- “ A finite set **CL** of clauses is/not satisfiable”  
is a **decidable statement**.
- **CL** has a **n** propositional variables ,hence we have  **$2^n$**  possible truth assignments **v** to examine and we can check them all by Truth Tables.
- This is called **Semantical Decidability**
- Problem: Exponential complexity

# Syntactical Decidability Method: Resolution Deduction

- **Goal** : We want to show that a finite set **CL** of clauses is **unsatisfiable**.
- **Method** : **Resolution deduction** :
- **Start** with **CL**; apply a transformation rule called **Resolution** as long as it is possible.
- **If** you **get {}**, then answer is **Yes**, i.e. **CL** is **unsatisfiable**
- **If** you **never get {}**, then answer is **NO**, i.e **CL** is **satisfiable**.

# Resolution Completeness Theorem 1

- **Completeness of the Resolution:**
- **CL** is **unsatisfiable** iff we obtain the empty clause **{}** by a multiple use of the **Resolution Rule**
- Symbolically: **CL ⊢ {}** means:
- deduce **{}** from **CL** by resolution rule;
- prove **{}** from **CL** by resolution

# Resolution Completeness Theorem 1

$\models \text{CL}$  denotes **CL is a tautology**  
 $\models \text{CL}$  denotes **CL is unsatisfiable**

- **Completeness 1 of the Resolution:**

$\models \text{CL}$  iff  $\text{CL} \vdash \{\}$

Completeness for a certain proof system S

$\models A$  iff  $\vdash A$



# Refutation

- **Refutation:** proving the contradiction

In classical logic we have that:

a formula **A** is a **tautology** iff  **$\neg A$**  is a **contradiction**

Symbolically:

$$\models A \text{ iff } \models \neg A$$

Observe:

$$\models (A_1 \wedge \dots \wedge A_n \Rightarrow B) \text{ iff } \models (A_1 \wedge \dots \wedge A_n \wedge \neg B)$$

# Refutation

By Resolution Completeness Theorem this is (almost, i.e. we need clauses not formulas!) equivalent to

$$\models (A_1 \wedge \dots \wedge A_n \Rightarrow B) \text{ iff } (A_1 \wedge \dots \wedge A_n \wedge \neg B) \vdash \{\}$$

It means that

to prove **B** from **A<sub>1</sub>, ..., A<sub>n</sub>** we keep **A<sub>1</sub>, ..., A<sub>n</sub>**,  
ADD **¬B** to it and use the Resolution Rule.

If we get **{}**, we have proved **B**.

It is called a **proof by REFUTATION**; to prove **B** we start with **¬B** and if we get a contradiction **{}**, we have proved **B**.

# Formulas – Clauses

**Resolution works only for clauses!**

To use it we need to transform our formulas into clauses. i.e. we prove the following

## Theorem

For any formula  $A \in F$ , there is a set of clauses  $CL_A$  such that  $A$  is logically equivalent to the set of clauses  $CL_A$

$CL_A$  is called a clausal form of  $A$ .

We have good set of Rules for Automatic of Transformation of  $A$  into the set of clauses and we will study it as next step.

# Completeness

- **Resolution Completeness 2:**
- $\models A$  iff  $\text{CL}_{\neg A} \vdash \{\}$
- $\text{CL}_{\neg A}$  = clausal form of  $\neg A$ .
- **Resolution Proof of A definition:**  
 $\vdash_R A$  iff  $\text{CL}_{\neg A} \vdash \{\}$

**Resolution Completeness 2:**

$$\models A \text{ iff } \vdash_R A$$

# Resolution Rule: R

- $C_1(a)$  means: clause  $C_1$  contains a positive literal  $a$
- $C_2(\neg a)$  means: clause  $C_2$  contains a negative literal  $\neg a$

- **Resolution Rule: R** (Two Premises)

$C_1(a) : C_2(\neg a)$       **Resolve on a**

**$(C_1 - \{a\} \cup C_2 - \{\neg a\})$  <- Resolvent**

# Resolution Rule: R

- Clauses are SETS!
- $\{C_1, C_2\}$  Complementary Pair

$C_1 = \{a, b, c, \neg d\}$

$C_2 = \{\neg a, \neg b, d\}$

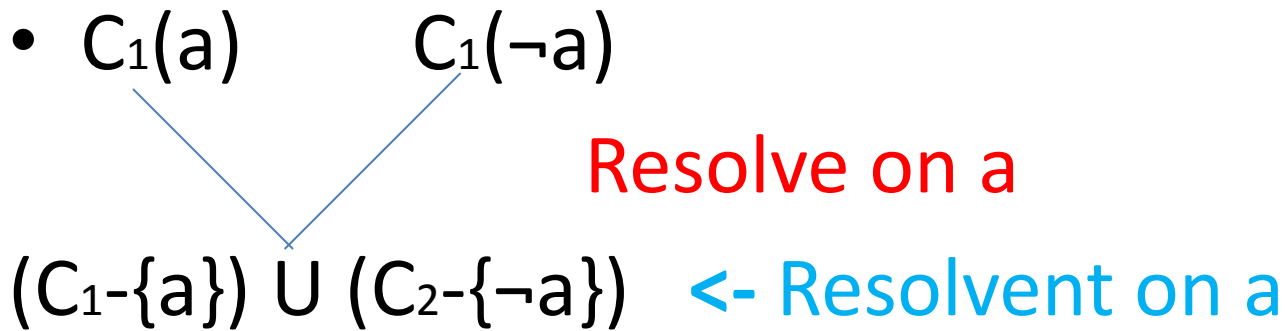
Resolve

on a

$\{b, c, \neg d, \neg b, d\}$  (Resolvent on a)

# Resolution Rule

- Resolution Rule takes 2 clauses and returns one. We usually write it in a form of a graph:
- **Definition:**  $C_1(a), C_1(\neg a)$  is a **Complementary Pair**



# Resolution Rule: R

- **CL** - set of clauses

Find all resolvents of **CL** means : locate all clauses in **CL** that are **Complementary Pairs** and **Resolve** them

$$C_1 = \{a, b, c, \neg d\}$$

$$C_2 = \{\neg a, \neg b, d\}$$

**CL** =  $\{C_1, C_2\}$  has **3 Complementary Pairs**

$$C_1(a), C_2(\neg a) - P1$$

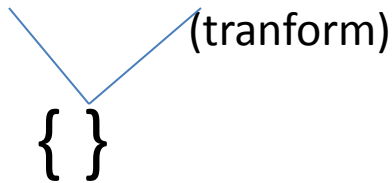
$$C_1(b), C_2(\neg b) - P2$$

$$C_2(d), C_1(\neg d) - P3$$



# Example

- $\{C_1(a) , C_2(\neg a)\}$



$$\{a\} \wedge \{\neg a\} = \phi \equiv \{ \}$$

$$C_1 = \{a , b , c , \neg d\}, C_2 = \{\neg a , \neg b , d\}$$

$$\{b , c , d\} \text{ (never } \{ \} \text{ from } \{C_1, C_2\})$$

- **Resolution Rule: R (Two Premises)**

$$\underline{C_1(a) : C_2(\neg a)}$$

**Resolve on a**

$$(C_1 - \{a\} \cup C_2 - \{\neg a\}) \leftarrow \text{Resolvent}$$

# Example

- $C_1 = \{a, b, c, \neg d\}$                        $C_2 = \{\neg a, \neg b, d\}$
- **CL** =  $\{C_1, C_2\} = \{C_2, C_1\}$  we have more than 1 resolvent!
- **Resolve on a:** We get  $\{b, c, \neg d, \neg a, d\}$
- **Resolve on b:** We get  $\{a, c, \neg d, \neg a, d\}$
- **Resolve on d:** We get  $\{a, b, c, \neg a, \neg b\}$

All resolvents of **CL**

# Example

- $\mathbf{CL} = \{C_1, C_2\} = \{C_2, C_1\}$

$C_1 = \{a, b, c, \neg d\}$

$C_2 = \{\neg a, \neg b, d\}$

Remember: **Resolution Rule uses one literal at the time!**

$C_1(a); C_2(\neg a)$  **Resolve on a** : we get  $\{b, c, \neg d, \neg a, d\}$

$C_1(b); C_2(\neg b)$  **Resolve on b** : we get  $\{a, c, \neg d, \neg a, d\}$

$C_1(d); C_2(\neg d)$  **Resolve on d** : we get  $\{a, b, c, \neg a, \neg b\}$

# Example

- We can also resolve PAIR P2 on a

$\{a, b, c, \neg d\}; \{\neg a, \neg b, d\}$   $\{C_1 C_2\}$

Resolve on a

$\{b, c, \neg d, \neg b, d\}$

These are **all** resolvent of pair P2.

# Example

$C_1(d) : C_2(\neg d)$  on Pair **P3**

$(C_1 - \{d\}) \cup (C_2 - \{\neg d\})$

$\{a, b, c, \neg d\} ; \{\neg a, \neg b, d\}$

Resolve on **d**

$\{a, b, c, \neg a, \neg b\}$

# Example

$C_1(b) : C_2(\neg b)$

Pair P2  $\{C_1 C_2\}$

$(C_1 - \{b\}) \cup (C_2 - \{\neg b\})$

$\{a, b, c, \neg d\} ; \{\neg a, \neg b, d\}$

Resolve on b

$\{a, c, \neg d, \neg a, d\} \leftarrow$  Resolvent on b

# Example

$C_1 = \{a, b, c, \neg d\}$  ;  $C_2 = \{\neg a, \neg b, c, d\}$

Resolve on  $b$

$\{a, c, \neg d, \neg a, d\}$

Resolvent on  $b$

Two clauses can have more than one resolvent  
(one complementary pair) – you can also  
resolve  $C_1 C_2$  on  $d$

# Resolution Deduction

- **CL** - set of clauses                      Deduce **C** from **CL**
- **CL**  $\vdash_R$  **C**

## Procedure:

**Start** with **CL** , apply the resolution rule **R** to **CL**

**Add** resolvent to **CL** (Data base) and

**Repeat** adding resolvents to already obtained Data base

**until** you get **C**.

**CL** =  $\{\{ a, b\}, \{ \neg a, c\}, \{ \neg b, c\}\}$

R on a     $\{b, c\}$

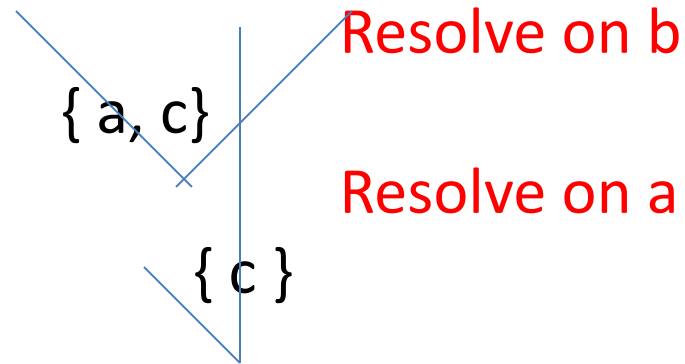
R on b     $\{ c \}$

**CL**  $\vdash_R$   $\{c\}$



# Example

- $\mathbf{CL} = \{\{ a, b \}, \{ \neg a, c \}, \{ \neg b, c \}\}$

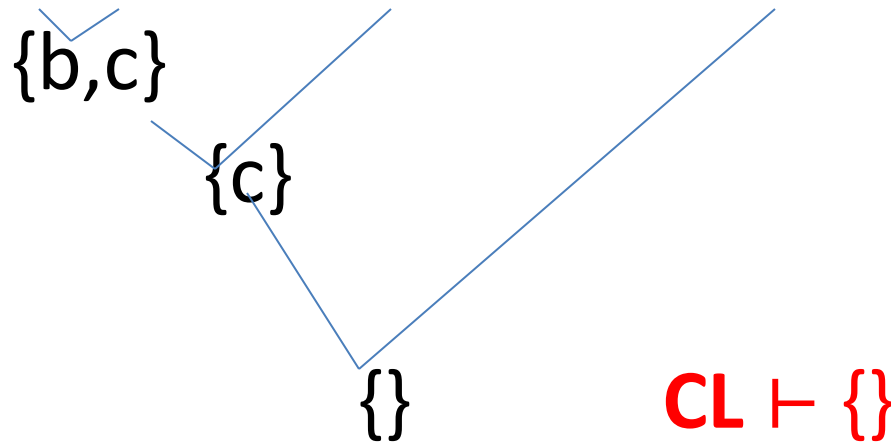


We have 2 possible deduction of  $\{ c \}$  from  $\mathbf{CL}$

$$\mathbf{CL} \vdash_R \{ c \}$$

# Example

- $\mathbf{CL} = \{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}, \{\neg c\}\}$



**CL is unsatisfiable** by completeness theorem :

$\models \mathbf{CL}$  iff  $\mathbf{CL} \vdash \{\}$

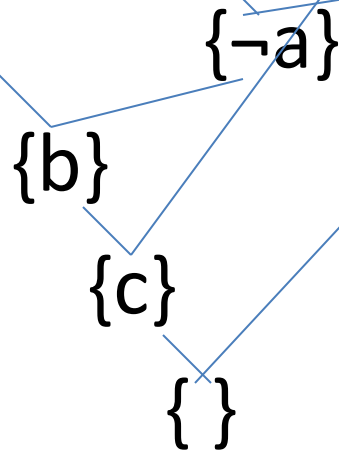
**Resolution deduction is not unique!**

–see another on next slide.

**Next:** Strategies for Resolution

# Example

- **CL** =  $\{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}, \{\neg c\}\}$



Another deduction of  $\{\}$  from **CL**.

# Exercise

- Let  $\mathbf{CL} = \{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}\}$

Find all possible deduction from  $\mathbf{CL}$

**Remember:**

1. If you get  $\{\}$ , it means  $\mathbf{CL}$  is **unsatisfiable**.
2. If you never get  $\{\}$ , it means  $\mathbf{CL}$  is **satisfiable**.

1 and 2 is true by **Completeness Theorem**

$$= | \mathbf{CL} \quad \text{iff} \quad \mathbf{CL} \vdash \{\}$$

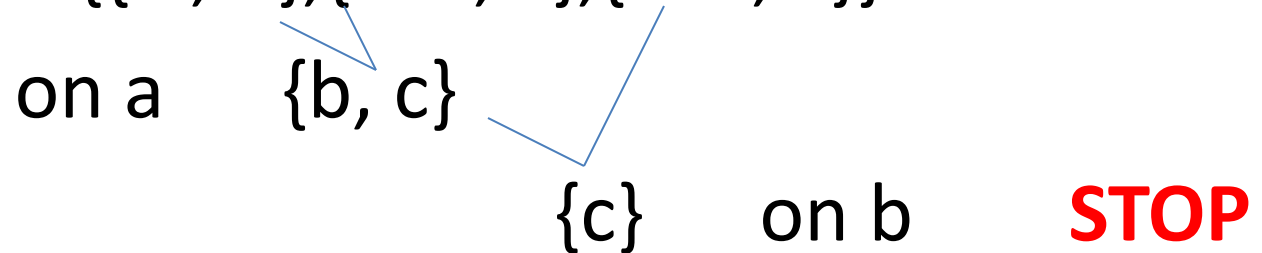
$\mathbf{CL}$  is unsatisfiable iff there is a deduction of  $\{\}$  from it.

$\mathbf{CL}$  is satisfiable iff there is NO deduction of  $\{\}$  From it.

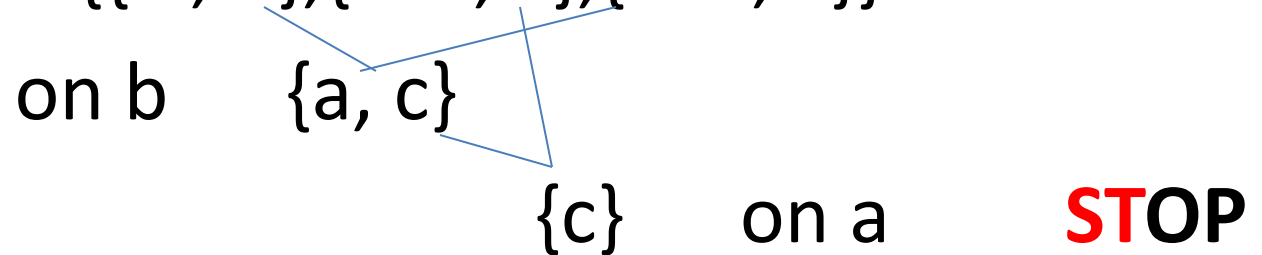
# Exercise

- **CL** =  $\{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}\}$

**Derivation 1:**  $\{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}\}$



**Derivation 2:**  $\{\{a, b\}, \{\neg a, c\}, \{\neg b, c\}\}$



**No more (possible) Derivations, i.e. by  
Completeness Theorem**

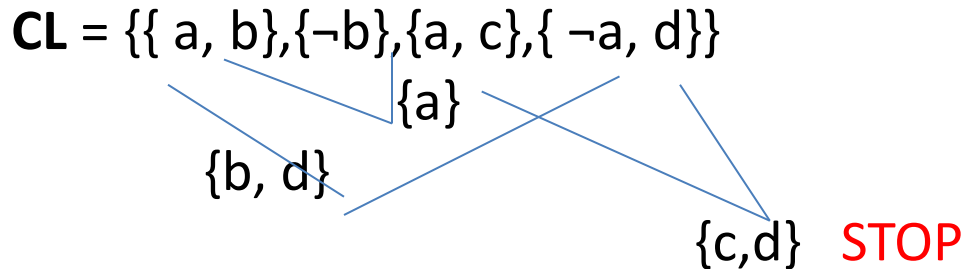
**CL is satisfiable**

# Exercise

- **CL** is unsatisfiable iff there is deduction of  $\{\}$  from it, i.e.

$$\mathbf{CL} \vdash_R \{\}$$

**CL** is satisfiable iff  $\mathbf{CL} \vdash_R \{\}$  (must cover all possibilities of deduction)



This is one derivation.

You must consider **ALL possible** derivations and show that none ends with  $\{\}$  to prove that **CL** is **satisfiable**.

# Exercise

- **Given:**  $CL = \{C_1, C_2, C_3, C_4\}$

$$CL = \{\{a, b, \neg b\}, \{\neg a, \neg b, d\}, \{a, b, \neg c\}, \{\neg a, c, b, e\}\}$$

1. **Find all complementary pairs** . Here they are:

$$\{C_1, C_2\} \{C_1, C_4\} ,$$

$$\{C_3, C_2\} \{C_2, C_3\} ,$$

$$\{C_3, C_4\} , \{C_2, C_4\}$$

2. **Find all resolvents** for your complementary pairs.

For example:  $C_1 = \{a, b, \neg b\}$  ,  $C_2 = \{\neg a, \neg b, d\}$  has 2 resolvents.

Resolve on a:  $\{\neg b, d, b\}$

Resolve on b;

$$\{a, \neg a, d, \neg b\}$$

# Exercise

- **CL** =  $\{C_1, C_2\}$ , for  $C_1 = \{a, b, c, \neg d\}$ ,  $C_2 = \{\neg a, \neg b, d\}$

**CL** has 3 resolvents :-

1.  $\{\neg a, \neg b, a, b, c\}$  – resolve on  $d$
2.  $\{\neg a, c, \neg d, d, a\}$  – resolve on  $b$
3.  $\{b, c, \neg d, d\}$  – resolve on  $a$

**Let now** **CL** =  $\{C_1, C_2, C_3\}$ ,  $C_1 = \{a\}$ ,  $C_2 = \{b, \neg a\}$ ,  
 $C_3 = \{\neg b, \neg a\}$

**Exercise:**

Find all Complementary Pairs + find all their resolvents



# Exercise Solution

**CL** contains 3 Complementary Pairs, each has one resolvent.

$\{a\}$     $\{b, \neg a\}$   
     $\{b\}$    resolve on  $a$

$\{a\}$     $\{\neg b, \neg a\}$   
     $\{\neg b\}$    resolve on  $a$

$\{b, \neg a\}$     $\{\neg b, \neg a\}$   
     $\{\neg a\}$    resolve on  $b$

**Complementary Pair:**

$C_1(x)$  ;  $C_2(\neg x)$