The Pumping Lemma for Regular Languages
Nonregular languages

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- Because the number of 0s isn’t limited, the machine needs to keep track of an unlimited number of possibilities.

- This cannot be done with any finite number of states.
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Language nonregularity

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If we can show that a language $L$ does not have this property we are guaranteed that $L$ is not regular.
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Consequence: A language may not be regular and still have strings that have all the properties of regular languages.
All strings in the language can be “pumped” if they are at least as long as a certain value, called the pumping length.
Pumping property

All strings in the language can be “pumped" if they are at least as long as a certain value, called the pumping length.

Meaning: each such string in the language contains a section that can be repeated any number of times with the resulting string remaining in the language.
Theorem 1.70

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2. Then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:
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  2. $|y| > 0$
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  2. $|y| > 0$
  3. $|xy| \leq p$
Interpretation

• **Recall that** \(|s|\) **represents** the **length of string** \(s\) **and** \(y^i\) **means that** \(y\) **may be concatenated** \(i\) **times**, **and** \(y^0 = \epsilon\)
Interpretation

- Recall that $|s|$ represents the length of string $s$ and $y^i$ means that $y$ may be concatenated $i$ times, and $y^0 = \epsilon$

- When $s = xyz$, either $x$ or $z$ may be $\epsilon$, but $y \neq \epsilon$
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- Recall that $|s|$ represents the length of string $s$ and $y^i$ means that $y$ may be concatenated $i$ times, and $y^0 = \epsilon$
- When $s = xyz$, either $x$ or $z$ may be $\epsilon$, but $y \neq \epsilon$
- Without condition $y \neq \epsilon$ theorem would be trivially true
Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that recognizes $A$. 

Assign a pumping length to be the number of states of $M$. Show that any string, maybe broken into three pieces satisfying the pumpinglemma's conditions, becomes vacuously true because all three conditions hold for all strings of length at least if there are no such strings.
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Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that recognizes $A$

- Assign a pumping length $p$ to be the number of states of $M$
- Show that any string $s \in A, |s| \geq p$ may be broken into three pieces $xyz$ satisfying the pumping lemma’s conditions
More ideas

- If $s \in A$ and $|s| \geq p$, consider a sequence of states that $M$ goes through to accept $s$, example: $q_1, q_3, q_{20}, \ldots, q_{13}$.
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• Since $M$ accepts $s$, $q_{13}$ must be final; if $|s| = n$ then the length of $q_1, q_3, q_{20}, \ldots, q_{13}$ is $n + 1$
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- Because $|s| = n$ and $|s| \geq p$ it result that $n + 1 > p$. 

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The sequence $q_1, q_3, q_{20}, \ldots, q_{13}$ must contain a repeated state, see Figure 1
Recognition sequence

\[ s = s_1 \uparrow s_2 \uparrow s_3 \uparrow s_4 \uparrow s_5 \uparrow s_6 \uparrow \cdots \]

\[ q_1 \quad q_3 \quad q_{20} \quad q_9 \quad q_{17} \quad q_9 \quad q_6 \quad q_{35}q_{13} \]

Figure 1: State \( q_9 \) repeats when \( M \) reads \( s \)
More ideas, continuation

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Note

The division specified above satisfies the 3 conditions.
Observations

Suppose that we run $M$ on $xyyz$
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- **Condition 1:** it is obvious that $M$ accepts $xyz$, $xyyz$, and in general $xy^i z$ for all $i > 0$. For $i = 0$, $xy^i z = xz$ which is also accepted because $z$ takes $M$ to $q_{13}$
Observations

Suppose that we run \( M \) on \( xyyz \)

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- **Condition 2:** Since \( |s| \geq p \), state \( q_9 \) is repeated. Then because \( y \) is the part between two successive occurrences of \( q_9 \), \( |y| > 0 \).
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- **Condition 2**: Since $|s| \geq p$, state $q_9$ is repeated. Then because $y$ is the part between two successive occurrences of $q_9$, $|y| > 0$.

- **Condition 3**: makes sure that $q_9$ is the first repetition in the sequence. Then by pigeonhole principle, the first $p + 1$ states in the sequence must contain a repetition. Therefore, $|xy| \leq p$
Pumping lemma’s proof

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that has $p$ states and recognizes $A$. Let $s = s_1s_2\ldots s_n$ be a string over $\Sigma$ of length $n \geq p$. Let $r_1, r_2, \ldots, r_{n+1}$ be the sequence of states while processing $s$, i.e., $r_{i+1} = \delta(r_i, s_i), 1 \leq i \leq n$. 
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- $n + 1 \geq p + 1$ and among the first $p + 1$ elements in $r_1, r_2, \ldots, r_{n+1}$ two must be the same state, say $r_j = r_k$. 


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- Because \( r_k \) occurs among the first \( p + 1 \) places in the sequence starting at \( r_1 \), we have \( k \leq p + 1 \)
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- $n + 1 \geq p + 1$ and among the first $p + 1$ elements in $r_1, r_2, \ldots, r_{n+1}$ two must be the same state, say $r_j = r_k$.
- Because $r_k$ occurs among the first $p + 1$ places in the sequence starting at $r_1$, we have $k \leq p + 1$
- Now let $x = s_1 \ldots s_{j-1}$, $y = s_j \ldots s_{k-1}$, $z = s_k \ldots s_n$. 
• As $x$ takes $M$ from $r_1$ to $r_j$, $y$ takes $M$ from $r_j$ to $r_j$, and $z$ takes $M$ from $r_j$ to $r_{n+1}$, which is an accept state, $M$ must accept $x y^i z$, for $i \geq 0$.
Note

- As $x$ takes $M$ from $r_1$ to $r_j$, $y$ takes $M$ from $r_j$ to $r_j$, and $z$ takes $M$ from $r_j$ to $r_{n+1}$, which is an accept state, $M$ must accept $x y^i z$, for $i \geq 0$

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- We know that $j \neq k$, so $|y| > 0$;
- We also know that $k \leq p + 1$, so $|xy| \leq p$. 
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- We know that $j \neq k$, so $|y| > 0$;

- We also know that $k \leq p + 1$, so $|xy| \leq p$

Thus, all conditions are satisfied and lemma is proven.
Note: To use this lemma we must also ensure that if the property stated by the pumping lemma is true then the language is regular.
Before using lemma

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**Proof:** assuming that each element of language $L$ satisfies the three conditions stated in pumping lemma we can easily construct a FA that recognizes $L$, that is, $L$ is regular.
Before using lemma

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Proof: assuming that each element of language $L$ satisfies the three conditions stated in pumping lemma we can easily construct a FA that recognizes $L$, that is, $L$ is regular.

Note: if only some elements of $L$ satisfy the three conditions it does not mean that $L$ is regular.
Using pumping lemma (PL)

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Using pumping lemma (PL)

Proving that a language $A$ is not regular using PL:

1. **Assume that $A$ is regular** in order to obtain a contradiction

2. **The pumping lemma guarantees** the existence of a pumping length $p$ s.t. all strings of length $p$ or greater in $A$ can be pumped

3. **Find** $s \in A$, $|s| \geq p$, that cannot be pumped: demonstrate that $s$ cannot be pumped by considering all ways of dividing $s$ into $x, y, z$, showing that for each division one of the pumping lemma conditions, (1) $xy^iz \in A$, (2) $|y| > 0$, (3) $|xy| \leq p$, fails.
Observations

• The existence of $s$ contradicts pumping lemma, hence $A$ cannot be regular
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• Finding $s$ sometimes takes a bit of creative thinking. Experimentation is suggested
Example 1: prove that $B = \{0^n 1^n | n \geq 0\}$ is not regular
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Assume that $B$ is regular and let $p$ be the pumping length of $B$. Choose $s = 0^p1^p \in B$; obviously $|0^p1^p| > p$. By pumping lemma $s = xyz$ such that for any $i \geq 0$, $xy^iz \in B$
Example, continuation

Consider the cases:

1. consists of s only. In this case has more s than s and so it is not in , violating condition 1.

2. consists of s only. This leads to the same contradiction.

3. consists of s and s. In this case may have the same number of s and s but they are out of order with some s before some s hence it cannot be in either .

The contradiction is unavoidable if we make the assumption that is regular so is not regular.
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Consider the cases:

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3. $y$ consists of 0s and 1s. In this case $xyyz$ may have the same number of 0s and 1s but they are out of order with some 1s before some 0s hence it cannot be in $B$ either.

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Example 2

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Prove that \( C = \{ w \mid w \text{ has an equal number of 0s and 1s} \} \) is not regular.

Proof: assume that \( C \) is regular and \( p \) is its pumping length. Let \( s = 0^p1^p \) with \( s \in C \). Then pumping lemma guarantees that \( s = xyz \), where \( xy^iz \in C \) for any \( i \geq 0 \).
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- Condition 3 states that $|xy| \leq p$, and in our case $xy = 0^p1^p$ and $|xy| > p$. Hence, $0^p1^p$ cannot be pumped.
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- If $|xy| \leq p$ then $y$ must consists of only 1s, so $x y y z \not\in C$ because there are more 1-s than 0-s.
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This gives us the **desired contradiction**.
Selecting $s = (01)^p$ leads us to trouble because this string can be pumped by the division: $x = \epsilon$, $y = 01$, $z = (01)^{p-1}$. Then $xy^i z \in C$ for any $i \geq 0$. 

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- But $C \cap 0^*1^* = \{0^n1^n | n \geq 0\}$ which is not regular.
- Hence, $C$ is not regular either.
Example 3

Show that $F = \{ww \mid w \in \{0, 1\}^*\}$ is nonregular using pumping lemma
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**Proof:** Assume that $F$ is regular and $p$ is its pumping length. Consider $s = 0^p10^p1 \in F$. Since $|s| > p$, $s = xyz$ and satisfies the conditions of the pumping lemma.
Note

• Condition 3 is again crucial because without it we could pump $s$ if we let $x = z = \epsilon$, so $xyyz \in F$.
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- **The string** $s = 0^p10^p1$ exhibits the essence of the nonregularity of $F$. 

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- **Condition 3 is again crucial** because without it we could pump \( s \) if we let \( x = z = \epsilon \), so \( xyyz \in F \).

- **The string** \( s = 0^p10^p1 \) **exhibits the essence** of the nonregularity of \( F \).

- **If we chose, say** \( 0^p0^p \in F \) **we fail** because this string can be pumped
Example 4

Show that $D = \{1^{n^2} | n \geq 0\}$ is nonregular.
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Proof by contradiction: Assume that $D$ is regular and let $p$ be its pumping length. Consider $s = 1^{p^2} \in D$, $|s| \geq p$.

Pumping lemma guarantees that $s$ can be split, $s = xyz$, where for all $i \geq 0$, $xy^iz \in D$
The elements of $D$ are strings whose lengths are perfect squares. Looking at first perfect squares we observe that they are: $0, 1, 4, 9, 25, 36, 49, 64, 81, \ldots$
Searching for a contradiction

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- Note the growing gap between these numbers: large members cannot be near each other
- Consider two strings $xy^i z$ and $xy^{i+1} z$ which differ from each other by a single repetition of $y$.
- If we chose $i$ very large the lengths of $xy^i z$ and $xy^{i+1} z$ cannot be both perfect square because they are too close to each other.
Turning this idea into a proof

Calculate the value of $i$ that gives us the contradiction.
Turning this idea into a proof

Calculate the value of \( i \) that gives us the contradiction.

- If \( m = n^2 \), calculating the difference we obtain
  \[
  (n + 1)^2 - n^2 = 2n + 1 = 2\sqrt{m} + 1
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Turning this idea into a proof

Calculate the value of $i$ that gives us the contradiction.

- If $m = n^2$, calculating the difference we obtain
  \[(n + 1)^2 - n^2 = 2n + 1 = 2\sqrt{m} + 1\]

- By pumping lemma $|xy^iz|$ and $|xy^{i+1}z|$ are both perfect squares. But letting $|xy^iz| = m$ we can see that they cannot be both perfect square if $|y| < 2\sqrt{|xy^iz|} + 1$, because they would be too close together.
Value of $i$ for contradiction

To calculate the value for $i$ that leads to contradiction we observe that:
Value of $i$ for contradiction

To calculate the value for $i$ that leads to contradiction we observe that:

- $|y| \leq |s| = p^2$
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- $|y| \leq |s| = p^2$
- Let $i = p^4$. Then
  \[
  |y| \leq p^2 = \sqrt{p^4} < 2\sqrt{p^4} + 1 \leq 2\sqrt{|xy^iz|} + 1
  \]
Example 5

Sometimes “pumping down" is useful when we apply pumping lemma.
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- We illustrate this using pumping lemma to prove that 
  \[ E = \{0^i1^j | i > j \} \] is not regular
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Sometimes “pumping down” is useful when we apply pumping lemma.

- We illustrate this using pumping lemma to prove that $E = \{0^i1^j|i > j\}$ is not regular.

- Proof: by contradiction using pumping lemma. Assume that $E$ is regular and its pumping length is $p$. 
Let $s = 0^{p+1}1^p$; From decomposition $s = xyz$, from condition 3, $|xy| \leq p$ it results that $y$ consists only of 0s.
Searching for a contradiction

- Let $s = 0^{p+1}1^p$; From decomposition $s = xyz$, from condition 3, $|xy| \leq p$ it results that $y$ consists only of 0s.
- Let us examine $xyyz$ to see if it is in $E$. Adding an extra-copy of $y$ increases the number of zeros. Since $E$ contains all strings $0^*1^*$ that have more 0s than 1s, it will still give a string in $E$.
Searching for a contradiction

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Try something else

- **Since** $xy^i z \in E$ even when $i = 0$, consider $i = 0$ and

$$xy^0 z = xz \in E.$$
Try something else

- **Since** $xy^i z \in E$ even when $i = 0$, consider $i = 0$ and $xy^0 z = xz \in E$.

- **This decreases the number of 0s in** $s$. 
Try something else

- Since $xy^iz \in E$ even when $i = 0$, consider $i = 0$ and $xy^0z = xz \in E$.
- This decreases the number of 0s in $s$.
- Since $s$ has just one more 0 than 1 and $xz$ cannot have more 0s than 1s,
  \[(xyz = 0^{p+1}1^p \text{ and } |y| \neq 0)\]
  $xz$ cannot be in $E$. 

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Try something else

- Since $xy^i z \in E$ even when $i = 0$, consider $i = 0$ and $xy^0 z = xz \in E$.
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  $(xyz = 0^{p+1} 1^p \text{ and } |y| \neq 0)$

  $xz$ cannot be in $E$.

This is the required contradiction
• **The pumping lemma says that** every regular language has a pumping length $p$, such that every string in the language of length at least $p$ can be pumped.
Minimum pumping length

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- Hence, if $p$ is a pumping length for a regular language $A$ so is any length $p' \geq p$.
Minimum pumping length

- The pumping lemma says that every regular language has a pumping length \( p \), such that every string in the language of length at least \( p \) can be pumped.
- Hence, if \( p \) is a pumping length for a regular language \( A \) so is any length \( p' \geq p \).
- The minimum pumping length for \( A \) is the smallest \( p \) that is a pumping length for \( A \).
Example

Consider $A = 01^*$. The minimum pumping length for $A$ is 2.
Example

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Reason: the string $s = 0 \in A$, $|s| = 1$ and $s$ cannot be pumped. But any string $s \in A$, $|s| \geq 2$ can be pumped because for $s = xyz$ where $x = 0$, $y = 1$, $z = rest$ and $xy^iz \in A$. Hence, the minimum pumping length for $A$ is 2.
Problem 1

Find the minimum pumping length for the language $0001^*$. 
Problem 1

Find the minimum pumping length for the language $0001^*$. 

Solution: The minimum pumping length for $0001^*$ is 4.

Reason: $000 \in 0001^*$ but $000$ cannot be pumped. Hence, 3 is not a pumping length for $0001^*$. If $s \in 0001^*$ and $|s| \geq 4$ $s$ can be pumped by the division $s = xyz$, $x = 000$, $y = 1$, $z = rest$. 
Problem 2

Find the minimum pumping length for the language $0^*1^*$. 

Solution:
The minimum pumping length of $0^*1^*$ is 1.

Reason: the minimum pumping length for $0^*1^*$ cannot be 0 because $0^*1^*$ is in the language but cannot be pumped. Every nonempty string $x$ can be pumped by the division: $x = uvw$, first character of $u$ and the rest of $x$. 

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Problem 2

Find the minimum pumping length for the language $0^*1^*$.

**Solution:** The minimum pumping length of $0^*1^*$ is 1.
Problem 2

Find the minimum pumping length for the language $0^*1^*$.

Solution: The minimum pumping length of $0^*1^*$ is 1.

Reason: The minimum pumping length for $0^*1^*$ cannot be 0 because $\epsilon$ is in the language but cannot be pumped. Every nonempty string $s \in 0^*1^*$, $|s| \geq 1$ can be pumped by the division: $s = xyz$, $x = \epsilon$, $y$ first character of $s$ and $z$ the rest of $s$. 

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Problem 3

Find the minimum pumping length for the language

$$0^*1^+0^+1^* \cup 10^*1.$$
Problem 3

Find the minimum pumping length for the language $0^*1^+0^+1^* \cup 10^*1$.

Solution: The minimum pumping length for $0^*1^+0^+1^* \cup 10^*1$ is 3.
Problem 3

Find the minimum pumping length for the language $0^*1^+0^+1^* \cup 10^*1$.

Solution: The minimum pumping length for $0^*1^+0^+1^* \cup 10^*1$ is 3.

Reason: The pumping length cannot be 2 because the string 11 is in the language and it cannot be pumped. Let $s$ be a string in the language of length at least 3. If $s$ is generated by $0^*1^+0^+1^*$ we can write is as $s = xyz$, $x = \epsilon$, $y$ is the first symbol of $s$, and $z$ is the rest of the string. If $s$ is generated by $10^*1$ we can write it as $s = xyz$, $x = 1$, $y = 0$ and $z$ is the remainder of $s$. 