cse303
ELEMENTS OF THE THEORY OF COMPUTATION

Professor Anita Wasilewska

LECTURE 8

CHAPTER 2
FINITE AUTOMATA

4. Languages that are Not Regular
5. State Minimization

CHAPTER 2
PART 4: Languages that are not Regular

Finite Automata and Regular Languages

Short Review

Finite Automata and Regular Languages

Finite Automata CLOSURE THEOREM

Finite Automata and Regular Languages MAIN THEOREM

Regular Languages CLOSURE THEOREM

Automata Closure Theorem

In order to prove the MAIN THEOREM that establishes a relationship between Finite Automata and Regular languages proved and used the following

Automata CLOSURE THEOREM

The class of languages accepted by **Finite Automata** (FA) is **closed** under the following operations

1. union
2. concatenation
3. Kleene's Star
4. complementation
5. intersection

Observe that we used the term **Finite Automata** (FA) so in the **proof** we can choose a DFA or a NDFA, as we have already proved their **equivalency**

Automata - Languages Main Theorem

Automata - Languages MAIN THEOREM

A language L is regular if and only if it is accepted by a finite automaton, i.e.

A language L is regular if and only if there is a finite automaton M, such that

$$L = L(M)$$

Regular Languages Closure Theorem

Directly from the the Automata and Regular Languages **Main Theorem** and Automata **Closure Theorem** we get the following

Regular Languages Closure Theorem

The class of REGULAR languages

is closed under the following operations

1. union
2. concatenation
3. Kleene's Star
4. complementation
5. intersection

Regular and non-Regular Languages

Languages that are Not Regular

We know that there are **uncountably** many and exactly C of all languages over any alphabet $\Sigma \neq \emptyset$

We also know that there are only $\aleph_0$, i.e. **infinitely countably** many regular languages

It means that we have **uncountably** many and . exactly C languages that **are not** regular

Reminder

A language $L \subseteq \Sigma^*$ is **regular** if and only if there is a regular expression $r \in \mathcal{R}$ that represents L, i.e. such that

$$L = \mathcal{L}(r)$$

Regular or not Regular Languages

We look now at some simple examples of languages that might be, or not be **regular**

E1 The language $L_1 = a^* b^*$ is **regular** because is defined by a regular expression

E2 The language

$$L_2 = \{a^n b^n : \quad n \geq 0\} \subseteq L_1$$

is not **regular**

We will **prove** prove it using a very important theorem to be proved that is called **Pumping Lemma**

Regular or not Regular Languages

Intuitively we can see that

$$L_2 = \{a^n b^n : \quad n \geq 0\}$$

can't be regular as we can't construct a **finite automaton** accepting it

Such automaton would need to have something like a **memory** to store, count and compare the number of a's with the number of b's

Regular or not Regular Languages

We will define and study in Chapter 3 a new class of **automata** that would accommodate the "memory" problem

They are called **Push Down Automata**

We will **prove** that they accept a larger class of languages, called context free languages

Regular or not Regular Languages

E3 The language $L_3 = a^*$ is **regular** because is defined by a regular expression

E4 The language $L_4 = \{a^n : \ n \geq 0\}$ is **regular** because in fact $L_3 = L_4$

E5 The language $L_4 = \{a^n : \ n \in Prime\}$ is **not regular**
We will **prove** it using Pumping Lemma

E6 The language $L_6 = \{a^n : \ n \in EVEN\}$ is **regular**

because in fact $L_6 = (aa)^*$

E7 The language

$L_7 = \{w \in \{a,b\}^* : w$ has an equal number of a' s and b's $\}$

is **not regular**

Proof

Assume that L_7 is **regular**

We know that $L_1 = a^*b^*$ is regular

Hence the language $L = L_7 \cap L_1$ is regular, as the class of regular languages is closed under intersection

But obviously, $L = \{a^nb^n : n \in N\}$ and was proved to be not regular

This **contradiction** proves that L_7 is **not regular**

Regular aor not Regular Languages

E8 The language $L_8 = \{ww^R : \; w \in \{a,b\}^*\}$ is **not regular**

We prove it using Pumping Lemma

E9 The language $L_9 = \{ww : \; w \in \{a,b\}^*\}$ is **not regular**

We prove it using Pumping Lemma

Regular or not Regular Languages

E10 The language $L_{10} = \{wcw : \quad w \in \{a, b\}^*\}$

is **not regular**

We prove it using Pumping Lemma

E11 The language $L_{11} = \{w\overline{w} : \quad w \in \{a, b\}^*\}$

where $\overline{w}$ stands for w with each occurrence of a is replaced by b, and vice versa

is **not regular**

We prove it using Pumping Lemma

Regular or not Regular Languages

E12 The language

$$L_{12} = \{xy \in \Sigma^* : \ x \in L \text{ and } y \notin L \text{ for any } \textbf{regular} \ L \subseteq \Sigma^*\}$$

is **regular**

Proof Observe that $L_{12} = L \circ \overline{L}$ where $\overline{L}$ denotes a complement of L, i.e.

$$\overline{L} = \{w \in \Sigma^* : \quad w \in \Sigma^* - L\}$$

L is **regular**, and so is $\overline{L}$, and $L_{12} = L \circ \overline{L}$ is **regular** by the following, already already proved theorem

Closure Theorem The class of languages accepted by Finite Automata FA is **closed** under $\cup, \cap, -, \circ, ^*$

E13 The language

$$L_{13} = \{w^R : \quad w \in L \quad \text{and} \quad L \quad \text{is regular} \quad \}$$

is **regular**

Definition For any language L we call the language

$$L_R = \{w^R : \quad w \in L\}$$

the reverse language of L

The **E13** says that the following holds

Fact

For any **regular** language L, its reverse language L^R is **regular**

Regular or not Regular Languages

Fact

For any **regular** language L, its reverse language L^R is **regular**

Proof Let $M = (K, \Sigma, \Delta, s, F)$ be such that L = L(M)

The reverse language L^R is accepted by a finite automata

$$M^R = (K \cup s', \Sigma, \Delta', s', F = \{s\})$$

where $s' \notin K$ and

$$\Delta' = \{(r, w, p) : \quad (p, w, r) \in \Delta, \quad w \in \Sigma^*\} \cup \{(s', e, q) : \ q \in F\}$$

We used the Lecture Definition of M

Regular and NOT Regular Languages

Proof of **E13** pictures

Diagram of M

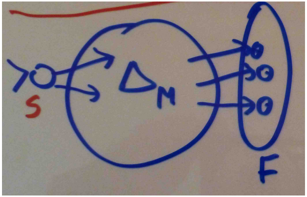
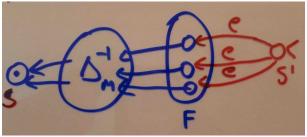


Diagram of M^R



E14

Any finite language is regular

Proof Let $L \subseteq \Sigma^*$ be a finite language , i.e.

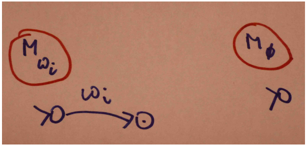
$$L = \emptyset \text{ or } L = \{w_1, w_2, \dots w_n\} \text{ for } n > 0\}$$

We construct the finite automata M such that

$$L(M) = L = \{w_1\} \cup \{w_2\} \cup \dots \{w_n\} = L_{w_1} \cup \dots \cup L_{w_n}$$

as $M = M_{w_1} \cup \dots \cup M_{w_n} \cup M_\emptyset$

where



Exercises

Exercise 1

Show that the language

$$L = \{xyx^R : \quad x, y \in \Sigma\}$$

is **regular** for any Σ

Exercises

Exercise 1

Show that the language

$$L = \{xyx^R : \quad x, y \in \Sigma\}$$

is **regular** for any Σ

Proof

For any $x \in \Sigma, x^R = x$

Σ is a finite set, hence

$$L = \{xyx : \; x, y \in \Sigma\}$$

is also finite and we just proved that any finite language is **regular**

Exercises

Exercise 2

Show that the class of regular languages **is not closed** with respect to subset relation.

Exercise 3

Given L_1, L_2 regular languages, is $L_1 \cap L_2$ also a regular language?

Exercise 2

Show that the class of regular languages **is not closed** with respect to subset relation.

Solution

Consider two languages

$$L_1 = \{a^n b^n : \quad n \in N\} \quad \text{and} \quad L_2 = a^* b^*$$

Obviously, $L_1 \subseteq L_2$ and L_1 is a **non-regular** subset of a regular L_2

Exercise 3

Given L_1, L_2 regular languages, is $L_1 \cap L_2$ also a regular language?

Solution

YES, it is because the class of regular languages is closed under $\cap$

Exercises

Exercise 4

Given L_1, L_2, such that $L_1 \cap L_2$ is a regular language

Does it imply that both languages L_1, L_2 must be regular?

Exercises

Exercise 4

Given L_1, L_2 , such that $L_1 \cap L_2$ is a regular language

Does it imply that both languages L_1, L_2 must be regular?

Solution

NO, it doesn't. Take the following L_1, L_2

$$L_1 = \{a^n b^n : n \in N\} \quad \text{and} \quad L_2 = \{a^n : n \in Prime\}$$

The language $L_1 \cap L_2 = \emptyset$ is a regular language none of L_1, L_2 is regular

Exercises

Exercise 5

Show that the language

$$L = \{xyx^R : \quad x, y \in \Sigma^*\}$$

is regular for any Σ

Exercise 5

Show that the language

$$L = \{xyx^R : \quad x, y \in \Sigma^*\}$$

is regular for any Σ

Solution

Take a case of $x = e \in \Sigma^*$

We get a language

$$L_1 = \{eye^R : e, y \in \Sigma^*\} \subseteq L$$

and of course $L_1 = \Sigma^*$ and so $\Sigma^* \subseteq L \subseteq \Sigma^*$

Hence $L = \Sigma^*$ and Σ^* is regular

This proves that L is regular

Exercises

Exercise 6

Given a regular language $L \subseteq \Sigma^*$

Show that the language

$$L_1 = \{xy \in \Sigma^* : \quad x \in L \quad \text{and} \quad y \notin L\}$$

is also regular

Exercises

Exercise 6

Given a regular language $L \subseteq \Sigma^*$

Show that the language

$$L_1 = \{xy \in \Sigma^* : \quad x \in L \quad \text{and} \quad y \notin L\}$$

is also regular

Solution

Observe that $L_1 = L \circ (\Sigma^* - L)$

L is regular, hence $(\Sigma^* - L)$ is regular (closure under complement), and so is L_1 by closure under concatenation

Pumping Lemma on Tests

Read Pumping Lemma statement and information about its role - you need to know it for **Midterm or Final**

The **proof** of the Pumping Lemma and its applications may be on the **Final**

Review Questions

Review Questions

Write SHORT answers

Q1

For any language $L \subseteq \Sigma^*$, $\Sigma \neq \emptyset$ there is a deterministic automata M, such that $L = L(M)$

Q2

Any regular language has a finite representation.

Q3

Any finite language is regular

Q4

Given L_1, L_2 languages over Σ, then $((L_1 \cap (\Sigma^* - L_2)) \cup L_2)L_1$ is a regular regular language

Review Questions

SHORT answers

Q1

For any language $L \subseteq \Sigma^*, \ \Sigma \neq \emptyset$ there is a deterministic automata M, such that $L = L(M)$

True only when L is regular

Q2

Any regular language has a finite representation.

True by definition of regular language and the fact that regular expression is a finite string

Q3

Any finite language is regular

True as we proved it

Q4

Given L_1, L_2 languages over Σ, then $((L_1 \cap (\Sigma^* - L_2)) \cup L_2)L_1$ is a regular regular language

True only when both are regular languages

Review Questions for Quiz

Write SHORT answers

Q5

For any finite automata M

$$L(M) = \bigcup \{R(1, j, n) : \quad q_j \in F\}$$

Q6

Σ in any Generalized Finite Automaton includes some regular expressions

Q7

Pumping Lemma says that we can always prove that a language is not regular

Q8

$L = \{a^n c^n : n \geq 0\}$ is regular

Review Questions

SHORT answers

Q5

For any finite automata M

$$L(M) = \bigcup \{R(1, j, n) : \quad q_j \in F\}$$

True only when M has n states and they are put in 1-1 sequence and $q_1 = s$

Q6

Σ in any Generalized Finite Automaton includes some regular expressions

True by definition

Review Questions

Q7

Pumping Lemma says that we can always prove that a language is not regular

Not True PL serves as a **tool** for proving that some languages are not regular

Q8

$L = \{a^n c^n : n \geq 0\}$ is regular

Not True we proved by PL that it is not regular

PUMPING LEMMA

Pumping Lemma

Pumping Lemma is one of a general class of Theorems called **pumping theorems**

They are called **pumping theorems** because they assert the existence of certain points in certain strings where a substring can be repeatedly inserted (pumping) without affecting the acceptability of the string

We present here two versions of the Pumping Lemma

First is the Lecture Notes version from the first edition of the Book and the second is the Book version (page 88) from the new edition

The Book version is a slight generalization of the Lecture version

Pumping Lemma 1

Let L be an infinite regular language over $\Sigma \neq \emptyset$

Then **there are** strings $x, y, z \in \Sigma^*$ such that

$$y \neq e \quad \text{and} \quad xy^n z \in L \ \text{ for all } \ n \geq 0$$

Observe that the Pumping Lemma 1 says that in an infinite regular language L, there is a word $w \in L$ that can be **re-written** as $w = xyz$ in such a way that $y \neq e$ and we "pump" the part y any number of times and still have that such obtained word is still in L, i.e. that $xy^n z \in L$ for all $n \geq 0$

Hence the name Pumping Lemma

Role of Pumping Lemma

We use the Pumping Lemma as a **tool** to carry **proofs** that some languages **are not regular**

Proof METHOD

Given an infinite language L we want to PROVE it to be NOT REGULAR

We proceed as follows

1. We assume that L is REGULAR

2.Hence by Pumping Lemma we get that there is a word $w \in L$ that can be **re-written** as $w = xyz,\ y \neq e$, and $xy^nz \in L$ for all $n \geq 0$

3. We examine the fact $xy^nz \in L$ for all $n \geq 0$

4. If we get a CONTRADICTION we have proved that the language L is **not regular**

Proof of Pumping Lemma 1

Pumping Lemma 1

Let L be an infinite regular language over $\Sigma \neq \emptyset$

Then **there are** strings $x, y, z \in \Sigma^*$ such that

$$y \neq e \quad \text{and} \quad xy^n z \in L \ \text{ for all } \ n \geq 0$$

Proof

Since L is regular, L is accepted by a deterministic finite automaton

$$M = (K, \ \Sigma, \ \delta, \ s, \ F)$$

Suppose that M has n states, i.e. $|K| = n$ for $n \geq 1$

Since L is **infinite**, M accepts some string $w \in L$ of length n or greater, i.e.

there is $w \in L$ such that $lw| = \mathbf{k} > \mathbf{n}$ and

$$w = \sigma_1\sigma_2 \ldots \sigma_k \qquad \text{for} \quad \sigma_i \in \Sigma, \quad 1 = 1, 2, \ldots, k$$

Proof of Pumping Lemma 1

Consider a **computation** of $w = \sigma_1\sigma_2\ldots\sigma_k \in L$:

$$(q_0, \sigma_1\sigma_2\ldots\sigma_k) \vdash_M (q_1, \sigma_2\ldots\sigma_k), \vdash_M$$

$$\ldots \ \ldots \vdash_M (q_{k-1}, \sigma_k), \vdash_M (q_k, e)$$

where q_0 is the initial state s of M and q_k is a final state of M

Since $|w| = \mathbf{k} > \mathbf{n}$ and M has only n states, by **Pigeon Hole Principle** we have that

there exist i and j, $0 \leq i < j \leq k$, such that $q_i = q_j$

That is, the string $\sigma_{i+1}\ldots\sigma_j$ is nonempty since $i + 1 \leq j$ and **drives** M from state q_i **back** to state q_i

But then this string $\sigma_{i+1}\ldots\sigma_j$ could be **removed** from w, or we could **insert** any number of its **repetitions** just after just after σ_j and M would still accept such string

Proof of Pumping Lemma 1

We just showed by **Pigeon Hole Principle** we have that M that accepts $w = \sigma_1\sigma_2\ldots\sigma_k \in L$ also **accepts** the string

$$\sigma_1\sigma_2\ldots\sigma_i(\sigma_{i+1}\ldots\sigma_j)^n\sigma_{j+1}\ldots\sigma_k \quad \text{for each} \quad n \geq 0$$

Observe that $\sigma_{i+1}\ldots\sigma_j$ is non-empty string since $i+1 \leq j$

That means that there exist strings

$$\mathbf{x} = \sigma_{\mathbf{1}}\sigma_{\mathbf{2}}\ldots\sigma_{\mathbf{i}}, \quad \mathbf{y} = \sigma_{\mathbf{i+1}}\ldots\sigma_{\mathbf{j}}, \quad \mathbf{z} = \sigma_{\mathbf{j+1}}\ldots\sigma_{\mathbf{k}} \quad \text{for} \quad y \neq e$$

such that

$$y \neq e \quad \text{and} \quad xy^nz \in L \quad \text{for all} \quad n \geq 0$$

Proof of Pumping Lemma 1

The computation of M that accepts $xy^n z$ is as follows

$$(q_o, xy^n z) \vdash_M{}^* (q_i, y^n z\ ) \vdash_M{}^* (q_i, y^{n-1} z\ )$$

$$\vdash_M{}^* \ \ldots \ \vdash_M{}^* (q_i, y^{n-1} z\ ) \vdash_M{}^* (q_k, e)$$

This **ends** the proof

Observe that the proof of the holds for **any word** $w \in L$ with $|w| \geq n$, where n is the number of states of deterministic M that accepts L

We get hence another version of the **Pumping Lemma 1**

Pumping Lemma 2

Let L be an infinite regular language over $\Sigma \neq \emptyset$

Then **there is** an integer $n \geq 1$ such that for **any word** $w \in L$ with lengths greater then n, i.e. $|w| \geq n$ **there are** $x, y, z \in \Sigma^*$ such that w can be re-written as $w = xyz$ and

$$y \neq e \quad \text{and} \quad xy^iz \in L \text{ for all } \textit{naturalnumbers } i \geq 0$$

Proof

Since L is regular, it is accepted by a deterministic finite automaton M that has $n \geq 1$ **states**

This is our integer $n \geq 1$

Let w be **any word** in L such that $|w| \geq n$

Such **words exist** as L in infinite

The rest of the proof exactly the same as in case of **Pumping Lemma 1**

Pumping Lemma

We write the **Pumping Lemma 2** symbolically using quantifiers symbols as follows

Pumping Lemma 2

Let L be an **infinite regular** language over $\Sigma \neq \emptyset$

Then the following holds

$$\exists_{n\geq 1} \forall_{w\in L} \Big(|w| \geq n \Rightarrow$$

$$\exists_{x,y,z\in\Sigma^*} \left(w = xyz \cap y \neq e \cap \forall_{i\geq 0}(xy^i z \in L)\right)\Big)$$

Book Pumping Lemma

Book Pumping Lemma is a STRONGER version of the **Pumping Lemma 2**

It applies to any any regular language, not to an infinite regular language, as the **Pumping Lemmas 1, 2**

Book Pumping Lemma

Let L be a regular language over $\Sigma \neq \emptyset$

Then **there is** an integer $n \geq 1$ such that **any word** $w \in L$ with $|w| \geq n$ can be re-written as $w = xyz$ such that

$$y \neq e, \quad |xy| \leq n, \quad x, y, z \in \Sigma^* \quad \text{and} \quad xy^i z \in L \quad \text{for all} \quad i \geq 0$$

Proof The proof goes exactly as in the case of Pumping Lemmas 1, 2

Notice that from the proof of Pumping Lemma 1

$$x = \sigma_1 \sigma_2 \ldots \sigma_i, \quad z = \sigma_{j+1} \ \ldots \ \sigma_k\} \ \text{for} \ 0 \leq i < j \leq n$$

and so by definition $|xy| \leq n$ for n being the **number of states** of the deterministic M that **accepts** L

Book Pumping Lemma

We write the Pumping Lemma 2 symbolically using quantifiers symbols as follows

Book Pumping Lemma

Let L be a regular language over $\Sigma \neq \emptyset$

Then the following holds

$$\exists_{n\geq 1} \forall_{w\in L} \left(|w| \geq n \Rightarrow \right.$$

$$\left. \exists_{x,y,z\in\Sigma^*} (w = xyz \cap y \neq e \cap |xy| \leq n \cap \forall_{i\geq 0} (xy^i z \in L)) \right)$$

A natural question arises:

WHY the Book Pumping Lemma applies when L is a regular **finite** language?

When L is a regular **finite** language the Lecture Lemmas do not apply

Book Pumping Lemma

Let's look at an example of a finite, and hence a regular language

$$L = \{a, b, ab, bb\}$$

Observe that the condition

$$\exists_{n\geq 1} \forall_{w\in L} \; (\; |w| \geq n \quad \Rightarrow$$

$$\exists_{x,y,z\in\Sigma^*} (w = xyz \; \cap \; y \neq e \; \cap \; |xy| \leq n \; \cap \forall_{i\geq 0} (xy^i z \in L)) \;)$$

of the Book Pumping Lemma **holds** because there exists $n = 3$ such that the conditions becomes as follows

Book Pumping Lemma

Take $n = 3$, or any $n \geq 3$ we get statement:

$$\exists_{n=3} \forall_{w \in L} \ (\ |w| \geq 3 \ \Rightarrow$$

$$\exists_{x,y,z \in \Sigma^*} (w = xyz \ \cap \ y \neq e \ \cap \ |xy| \leq n \ \cap \forall_{i \geq 0} (xy^i z \in L)) \ )$$

Observe that the above is a TRUE statement because the statement $|w| \geq 3$ is FALSE for all $w \in L = \{a, b, ab, bb\}$

By definition, the implication FALSE $\Rightarrow$ ANYTHING is always TRUE, hence the whole statement is TRUE

Book Pumping Lemma

The same reasoning applies for any **finite** (and hence regular) language

In general, let L be any **finite** language

Let $m = max\{|w| : w \in L\}$

Such m **exists** because L is finite

Take $n = m + 1$ as the n in the condition of the Book Pumping Lemma

The Lemma condition is TRUE for **all** $w \in L$, because the statement

$|w| \geq m + 1$ is FALSE for **all** $w \in L$

By definition, the implication FALSE$\Rightarrow$ ANYTING is always TRUE, hence the whole statement is TRUE

Pumping Lemma Applications

Use Pumping Lemma to **prove** the following

Fact 1

The language $L \subseteq \{a, b\}^*$ defined as follows

$$L = \{a^n b^n : \quad n > 0\}$$

IS NOT regular

Obviously, L i infinite and we use the Lecture version

Pumping Lemma 1

Let L be an infinite regular language over $\Sigma \neq \emptyset$

Then **there are** strings $x, y, z \in \Sigma^*$ such that

$$y \neq e \quad \text{and} \quad xy^n z \in L \quad \text{for all} \quad n \geq 0$$

Pumping Lemma Applications

Reminder: we proceed as follows

1. We assume that L is REGULAR

2. Hence by Pumping Lemma we get that there is a word $w \in L$ that can be **re-written** as $w = xyz$ for $y \neq e$ and $xy^n z \in L$ for all $n \geq 0$

3. We examine the fact $xy^n z \in L$ for all $n \geq 0$

4. If we get a CONTRADICTION we have proved that L is NOT REGULAR

Pumping Lemma Applications

Assume that

$$L = \{a^m b^m : \quad m \geq 0\}$$

IS REGULAR

L is infinite hence Pumping Lemma 1 applies, so there is a word $w \in L$ that can be **re-written** as $w = xyz$ for $y \neq e$ and $xy^n z \in L$ for all $n \geq 0$

There are **three** possibilities for $y \neq e$

We will show that in **each case** we prove that $xy^n z \in L$ is impossible (contradiction)

Pumping Lemma Applications

Consider $w = xyz \in L$, i.e. $xyz = a^m b^m$ for some $m \geq 0$

We have to consider the following cases

Case 1

y consists entirely of a's

Case 2

y consists entirely of b's

Case 3

y contains both some a's followed by some b's

We will show that in each case assumption that $xy^n z \in L$ for all n leads to CONTRADICTION

Pumping Lemma Applications

Consider $w = xyz \in L$, i.e. $xyz = a^m b^m$ for some $m \geq 0$

Case 1: y consists entirely of a's

So x **must** consists entirely of a's only and z **must** consists of some a's followed by some b's

Remember that only we must have that $y \neq e$

We have the following situation

$x = a^p$ for $p \geq 0$ as x can be empty

$y = a^q$ for $q > 0$ as y must be nonempty

$z = a^r b^s$ for $r \geq 0$, $s > 0$ as we must have some b's

Pumping Lemma Applications

The condition $xy^n z \in L$ for all $n \geq 0$ becomes as follows

$$a^p(a^q)^n a^r b^s = a^{p+nq+r} b^s \in L$$

for all p, q, n, r, s such that the following conditions hold

$$\textbf{C1:} \quad p \geq 0, \quad q > 0, \quad n \geq 0, \quad r \geq 0, \quad s > 0$$

By definition of L

$$a^{p+nq+r} b^s \in L \quad \text{iff} \quad [p + nq + r = s$$

Take case: $p = 0, \ r = 0, \ q > 0, \ n = 0$

We get $s = 0$ CONTRADICTION with **C1:** $s > 0$

Pumping Lemma Applications

Consider $xyz = a^m b^m$ for some $m \geq 0$

Case 2: y consists of b's only

So x **must** consists of some a's followed by some b's and z **must** have only b's, possibly none

We have the following situation

$x = a^p b^r$ for $p > 0$ as y has at least one b and $r \geq 0$

$y = b^q$ for $q > 0$ as y must be nonempty

$z = b^s$ for $s \geq 0$

Pumping Lemma Applications

The condition $xy^n z \in L$ for all $n \geq 0$ becomes as follows

$$a^p b^r (b^q)^n b^s = a^p b^{r+nq+r} \in L$$

for all p, q, n, r, s such that the following conditions hold

C2: $p > 0, \; r \geq 0 \quad q > 0, \quad n \geq 0, \quad s \geq 0$

By definition of L

$$a^p b^{r+nq+r} \in L \quad \text{iff} \quad [p = r + qn + s$$

Take case: $r = 0, \quad n = 0, \quad q > 0$

We get $p = 0$ CONTRADICTION with **C2:** $p > 0$

Pumping Lemma Applications

Consider $xyz = a^m b^m$ for some $m \geq 0$

Case 3: y contains both a's and a's

So $y = a^p b^r$ for $p > 0$ and $r > 0$

Case $y = b^r a^p$ is impossible

Take case: $y = ab, \ x = e, \ z = e$ and $n = 2$

By Pumping Lemma we get that $y^2 \in L$

But this is a CONTRADICTION with $y^2 = abab \notin L$

We covered all cases and it **ends the proof**

Pumping Lemma Applications

Use Pumping Lemma to **prove** the following

Fact 2

The language $L \subseteq \{a\}^*$ defined as follows

$$L = \{a^n : \quad n \in Prime\}$$

IS NOT regular

Obviously, L i infinite and we use the Lecture version

Proof

Assume that L is regular, hence as L is infinite, so there is a word $w \in L$ that can be **re-written** as $w = xyz$ for $y \neq e$ and $xy^n z \in L$ for all $n \geq 0$

Consider $w = xyz \in L$, i.e. $xyz = a^m$ for some $m > 0$ and $m \in Prime$

Pumping Lemma Applications

Then

$$x = a^p, \quad y = a^q, \quad z = a^r \text{ for } \quad p \geq 0, \quad q > 0, \quad r \geq 0$$

The condition $xy^nz \in L$ for all $n \geq 0$ becomes as follows

$$a^p(a^q)^n a^r = a^{p+nq+r} \in L$$

It means that for all n, p, q, r the following condition hold

C $n \geq 0, \; p \geq 0, \; q > 0, \; r \geq 0,$ and $p + nq + r \in Prime$

But this is IMPOSSIBLE

Pumping Lemma Applications

Take n = p + 2q + r + 2 and **evaluate:**

$$p + nq + r = p + (p + 2q + r + 2)q + r =$$

$$p(1 + q) + 2q(q + 1) + r(q + 1) = (q + 1)(p + 2q + r)$$

By the above and the condition **C** we get that

$p + nq + r \in Prime$ and $p + nq + r = (q + 1)(p + 2q + r)$

and both factors are natural numbers greater than 1 what is a CONTRADICTION

This **ends the proof**