# cse303
# ELEMENTS OF THE THEORY OF COMPUTATION

Professor Anita Wasilewska

# LECTURE 7

# CHAPTER 2
# FINITE AUTOMATA

1. Deterministic Finite Automata DFA
2. Nondeterministic Finite Automata NDFA
3. Finite Automata and Regular Expressions
4. Languages that are Not Regular
5. State Minimization

# CHAPTER 2
# PART 3: Finite Automata and Regular Expressions

# Finite Automata and Regular Expressions

The goal of this part of chapter 2 is to prove a **theorem** that establishes a **relationship** between Finite Automata and Regular languages, i.e to **prove** that following

**MAIN THEOREM**

A language L is regular if and only if it is accepted by a finite automaton, i.e.

A language L is regular if and only if there is a finite automaton M, such that

$$L = L(M)$$

# Closure Theorem

To achieve our goal we first prove the following

**CLOSURE THEOREM**

The class of languages accepted by **Finite Automata** (FA) is **closed** under the following operations

1. union
2. concatenation
3. Kleene's Star
4. complementation
5. intersection

**Observe** that we used the term **Finite Automata** (FA) so in the **proof** we can choose a DFA or a NDFA, as we have already proved their **equivalency**

# Closure Theorem

**Remember** that languages are **sets**, so we have the set em[] operations $\cup$, $\cap$, $-$, defined for any $L_1, L_2 \subseteq \Sigma^*$, i.e the languages

$$L = L_1 \cup L_2, \quad L = L_1 \cap L_2, \quad L = \Sigma^* - L_1$$

We also defined the languages specific operations of concatenation and Kleene's Star , i.e. the

languages

$$L = L_1 \circ L_2 \quad \text{and} \quad L = {L_1}^*$$

# Closure Under Union

**1.** The class of languages accepted by Finite Automata (FA) is **closed** under union

**Proof**

Let $M_1,\ M_2$ be two NDFA finite automata

We **construct** a NDF automaton M, such that

$$L(M) = L(M_1) \cup L(M_2)$$

Let $M_1 = (K_1,\ \Sigma,\ \Delta_1,\ s_1,\ F_1)$ and

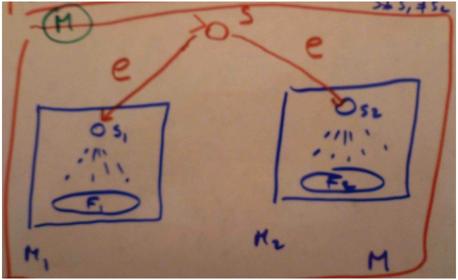
$M_2 = (K_2,\ \Sigma,\ \Delta_2,\ s_2,\ F_2)$

Where (we rename the states, if needed)

$\Sigma = \Sigma_1 \cup \Sigma_2, \quad s_1 \neq s_2, \quad K_1 \cap K_2 = \emptyset \quad F_1 \cap F_2 = \emptyset$

# Closure Under Union

We **picture** M, such that $L(M) = L(M_1) \cup L(M_2)$ as follows




M goes nondeterministically to $M_1$ or to $M_2$ reading nothing so we get

$$w \in L(M) \quad \text{if and only if} \quad w \in M_1 \quad \text{or} \quad w \in M_2$$

and hence

$$L(M) = L(M_1) \cup L(M_2)$$

# Closure Under Union

We **define formally**

$$M = M_1 \cup M_2 = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

where

$K = K_1 \cup K_2 \cup \{s\}$ for $s \notin K_1 \cup K_2$

$s$ is a **new** state and

$F = F_1 \cup F_2$, $\Delta = \Delta_1 \cup \Delta_2 \cup \{(s, e, s_1),\ (s, e, s_2)\}$

for $s_1$ - initial state of $M_1$ and

$s_2$ the initial state of $M_2$

**Observe** that by Mathematical Induction we construct,

for any $n \geq 2$ an automaton $M = M_1 \cup M_2 \cup \ \ldots\ M_n$ such that

$$L(M) = L(M_1) \cup L(M_2) \cup \ \ldots\ L(M_n)$$

**Formal proof**

Directly from the definition we get

$w \in L(M)$ if and only if

$\exists_q((q \in F = F_1 \cup F_2) \cap ((s, w) \vdash_M{}^*(q, e))$ if and only if

$\exists_q(((q \in F_1) \cup (q \in F_2)) \cap ((s, w) \vdash_M{}^*(q, e))$ if and only if

$\exists_q((q \in F_1) \cap ((s, w) \vdash_M{}^*(q, e)) \cup$

$\exists_q((q \in F_2) \cap ((s, w) \vdash_M{}^*(q, e)))$ if and only if

$w \in L(M_1) \cup w \in L(M_2)$, what proves that

$$L(M) = L(M_1) \cup L(M_2)$$

We used the following Law of Quantifiers

$$\exists_x(A(x) \cup B(x)) \equiv (\exists_x A(x) \cup \exists_x B(x))$$

# Examples

**Example 1**

**Diagram** of $M_1$ such that $L(M_1) = aba^*$ is

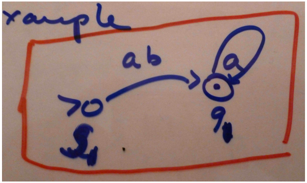
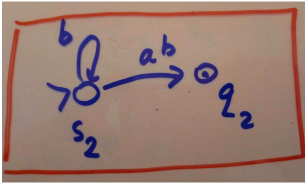



**Diagram** of $M_2$ such that $L(M_2) = b^*ab$ is




We construct $M = M_1 \cup M_2$ such that

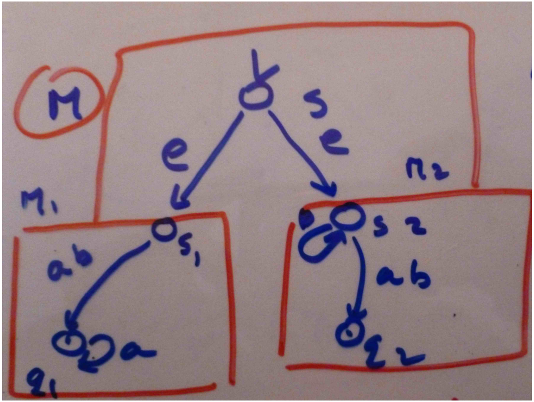
$$L(M) = aba^* \cup b^*ab = L(M_1) \cup L(M_2)$$

as follows

# Examples

**Example 1**

**Diagram** of $M$ such that $L(M) = aba^* \cup b^*ab$ is



**Example 2**

**Diagram** of $M_1$ such that $L(M_1) = b^*abc$ is

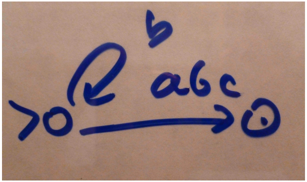
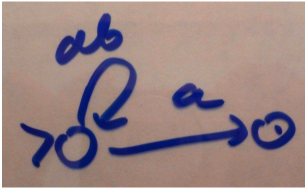



**Diagram** of $M_2$ such that $L(M_2) = (ab)^*a$ is



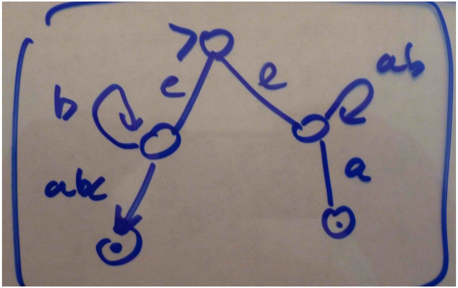

We construct $M = M_1 \cup M_2$ such that

$$L(M) = b^*abc \cup (ab)^*a = L(M_1) \cup L(M_2)$$

as follows

## Examples

**Diagram** of $M$ such that $L(M) = b^* abc \cup (ab)^* a$ is




This is a schema diagram

If we need to **specify** the components we put **names** on states on the diagrams

# Closure Under Concatenation

**2.** The class of languages accepted by Finite Automata is **closed** under concatenation

**Proof**

Let $M_1, M_2$ be two NDFA

We **construct** a NDF automaton M, such that

$$L(M) = L(M_1) \circ L(M_2)$$

Let $M_1 = (K_1, \Sigma, \Delta_1, s_1, F_1)$ and

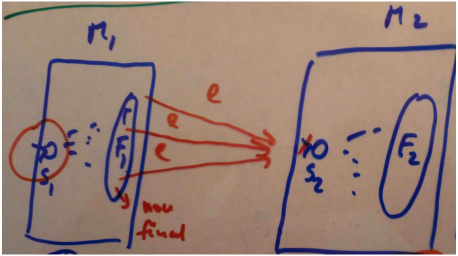
$M_2 = (K_2, \Sigma, \Delta_2, s_2, F_2)$

Where (if needed we re-name states)

$\Sigma = \Sigma_1 \cup \Sigma_2, \quad s_1 \neq s_2, \quad K_1 \cap K_2 = \emptyset \quad F_1 \cap F_2 = \emptyset$

# Closure Under Concatenation

We **picture** M, such that $L(M) = L(M_1) \circ L(M_2)$ as follows




The final states from $F_1$ of $M_1$ become **internal** states of M

The initial state $s_2$ of $M_2$ becomes an **internal** state of M

M goes nondeterministically from ex-final states of $M_1$ to the ex-initial state of $M_2$ **reading** nothing

# Closure Under Concatenation

We **define formally**

$$M = M_1 \circ M_2 = (K,\ \Sigma,\ \Delta,\ s_1,\ F_2)$$

where

$K = K_1 \cup K_2$

$s_1$ of $M_1$ is the initial state

$F_2$ of $M_2$ is the set of final states

$\Delta = \Delta_1 \cup \Delta_2 \cup \{(q, e, s_2) :$ for $q \in F_1\}$

Directly from the definition we get

$w \in L(M)$ iff $w = w_1 \circ w_2$ for $w_1 \in L_1,\ w_2 \in L_2$

and hence

$$L(M) = L(M_1) \circ L(M_2)$$

# Examples

**Diagram** of $M_1$ such that $L(M_1) = aba^*$ is

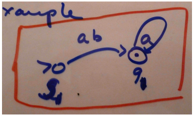
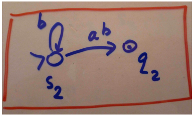


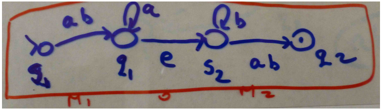

**Diagram** of $M_2$ such that $L(M_2) = b^*ab$ is




We construct $M = M_1 \circ M_2$ such that

$$L(M) = aba^* \circ b^*ab = L(M_1) \circ L(M_2)$$

as follows



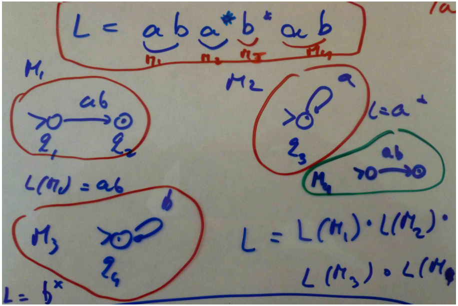
# Examples

Given a language $L = aba^*b^*ab$

**Observe** that we can reprezent L as, for example, the following concatenation

$$L = ab \circ a^* \circ b^* \circ ab$$

Then we construct "easy" automata $M_1, M_2, M_3, M_4$ as follows



# Examples

We know, by Mathematical Induction that we can construct, for any $n \geq 2$ an automaton

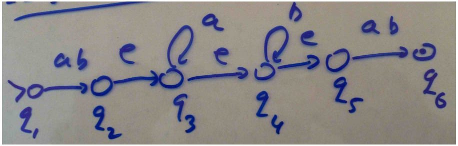
$$M = M_1 \circ M_2 \circ \ \circ M_n$$

such that

$$L(M) = L(M_1) \circ \ \ldots \ \circ \ L(M_n)$$

In our case n=4 and we get

**Diagram** of M



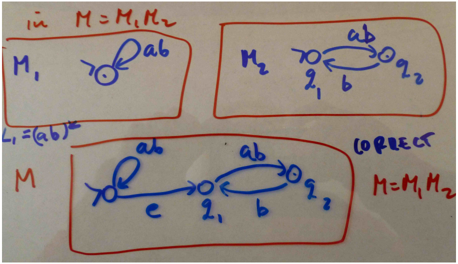

and $L(M) = aba^*b^*ab$

**Question**

Why we have to go be the transactions $(q, e, s_2)$ between $M_1$ and $M_2$ while constructing $M = M_1 \circ M_2$?

**Example** of a construction when we can't SKIP the transaction $(q, e, s_2)$

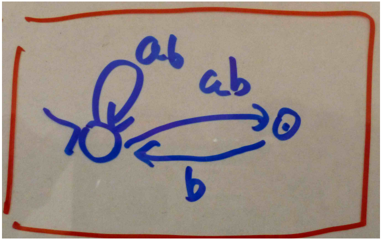
Here is a **correct** construction of $M = M_1 \circ M_2$




**Observe** that $abbabab \notin L(M)$

## Question

Here is a construction of $M' = M_1 \circ M_2$ without the transaction $(q, e, s_2)$




**Observe** that $abbabab \in L(M')$ and $abbabab \notin L(M)$

We hence proved that skipping the transactions $(q, e, s_2)$ between $M_1$ and $M_2$ leads to automata accepting different languages

# Closure Under Kleene's Star

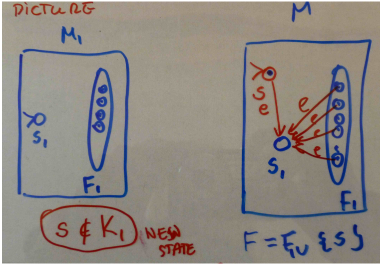
**3.** The class of languages accepted by Finite Automata is **closed** under Kleene's Star

**Proof** Let $M_1 = (K_1, \Sigma, \Delta_1, s_1, F_1)$

We **construct** a NDF automaton $M = M_1{}^*$, such that

$$L(M) = L(M_1)^*$$

Here is a **diagram**



# Closure Under Kleene's Star

Given $M_1 = (K_1,\ \Sigma,\ \Delta_1,\ s_1,\ F_1)$

We **define formally**

$$M = {M_1}^* = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

where

$K = K_1 \cup \{s\}$ for $s \notin K_1$

$s$ is new initial state, $s_1$ becomes an internal state

$F = F_1 \cup \{s\}$

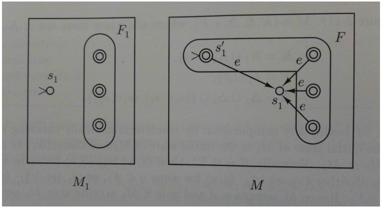
$\Delta = \Delta_1 \cup \{(s, e, s_1)\} \cup \{(q, e, s_1) :$ for $q \in F_1\}$

Directly from the definition we get

$$L(M) = L(M_1)^*$$

# Closure Under Kleene's Star

The Book **diagram** is




Given $M_1 = (K_1, \Sigma, \Delta_1, s_1, F_1)$

We define

$$M_1{}^* = (K_1 \cup \{s\}, \Sigma, \Delta, s, F_1 \cup \{s\})$$

where $s$ is a new initial state and

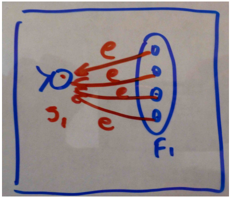
$\Delta = \Delta_1 \cup \{(s, e, s_1)\} \cup \{(q, e, s_1) :$ for $q \in F_1\}$

# Two Questions

Here **two questions** about the construction of $M = M_1{}^*$

**Q1** Why do we need to make the NEW initial state $s$ of $M$ also a FINAL state?

**Q2** Why can't SKIP the introduction of the NEW initial state and design $M = M_1{}^*$ as follows




**Q1 + Q2** give us answer why we construct $M = M_1{}^*$ as we did, i.e. provides the motivation for the correctness of the construction

# Question 1 Answer

**Observe** that the definition of $M = M_1{}^*$ must be correct for ALL automata $M_1$ and hence in particular for $M_1$ such that $F_1 = \emptyset$,

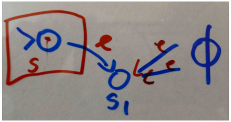
In this case we have that $L(M_1) = \emptyset$

But we know that

$$L(M) = L(M_1)^* = \emptyset^* = \{e\}$$

This proves that $M = M_1{}^*$ must accept e, and hence we must make $s$ of $M$ also a FINAL state

**Diagram**



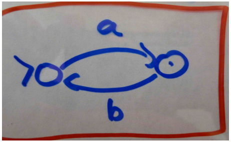
# Question 2 Answer

**Q2** Why can't SKIP the introduction of the NEW initial state and design $M = M_1{}^*$

Here is an **example**

Let $M_1$ , such that $L(M_1) = a(ba)^*$

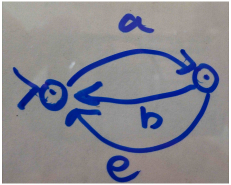
$M_1$ is defined by a **diagram**




$$L(M_1)^* = (a(ba)^*)^*$$

## Question 2 Answer

Here is a **diagram** of $M$ where we skipped the introduction of a new initial state



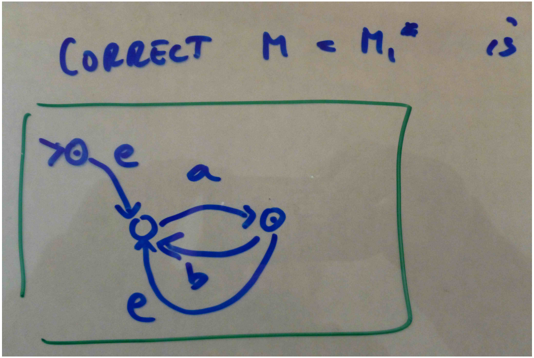

**Observe** that $ab \in L(M)$ , but

$$ab \notin (a(ba)^*)^* = L(M_1)^*$$

This proves **incorrectness** of the above construction

## Correct Diagram

The CORRECT **diagram** of $M = M_1{}^*$ is



**Exercise 1**

Construct M such that

$$L(M) = (ab^*ba \cup a^*b)^*$$

**Observe** that

$$L(M) = (L(M_1) \cup L(M_2))^*$$

and

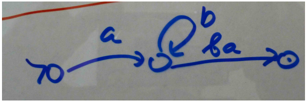
$$M = (M_1 \cup M_2)^*$$

# Exercise 1

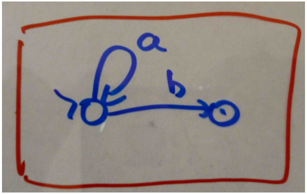
**Solution**

We construct M such that $L(M) = (ab^*ba \cup a^*b)^*$ in the following steps using the **Closure Theorem** definitions

**Step 1** Construct $M_1$ for $L(M_1) = ab^*ba$

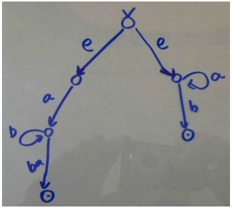



**Step 2** Construct $M_2$ for $L(M_2) = a^*b$

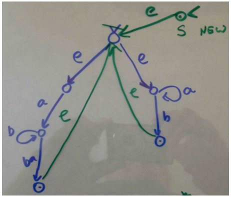


# Exercise

**Step 3** Construct $M_1 \cup M_2$




**Step 4** Construct $M = (M_1 \cup M_2)^*$



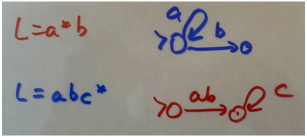

$$L(M) = (ab^*ba \cup a^*b)^*$$

**Exercise 2**

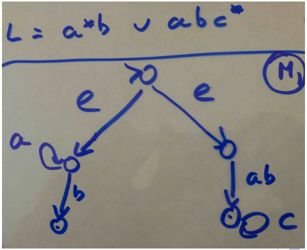
Construct M such that $L(M) = (a^*b \cup abc^*)a^*b^*$

**Solution** We construct M in the following steps using the **Closure Theorem** definitions

**Step 1** Construct $N_1, N_2$ for $L = a^*b$ and $L = abc^*$

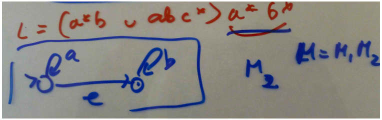



**Step 2** Construct $M_1 = N_1 \cup N_2$

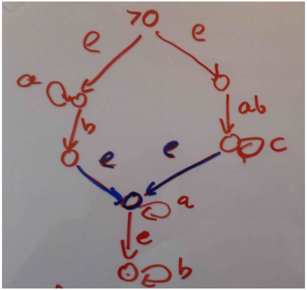


# Exercise 2

**Step 3** Construct $M_2$ for $L = a^*b^*$




**Step 4** Construct $M = (M_1 \circ M_2)^*$




$$L(M) = (a^*b \cup abc^*)a^*b^*$$

Back to Closure Theorem

**CLOSURE THEOREM**

The class of languages accepted by **Finite Automata** FA) is **closed** under the following operations

1. union **proved**
2. concatenation **proved**
3. Kleene's Star **proved**
4. complementation
5. intersection

**Observe** that we used the term **Finite Automata** (FA) so in the

proof we can choose a DFA or NDFA, as we have already proved their **equivelency**

# Closure Under Complementation

**4.** The class of languages accepted by Finite Automata is **closed** under complementation

**Proof** Let

$$M = (K,\ \Sigma,\ \delta,\ s,\ F)$$

be a **deterministic** finite automaton DFA

The complementary language $\overline{L} = \Sigma^* - L(M)$ is accepted by the DFA denoted by $\overline{M}$ that is identical with M except that final and nonfinal states are interchanged, i.e. we define

$$\overline{M} = (K,\ \Sigma,\ \delta,\ s,\ K - F)$$

and we have

$$L(\overline{M}) = \Sigma^* - L(M)$$

# Closure Under Intersection

**4.** The class of languages accepted by Finite Automata is **closed** under intersection

**Proof 1**

Languages are sets so we have have the following property

$$L_1 \cap L_2 = \Sigma^* - ((\Sigma^* - L_1) \cup (\Sigma^* - L_2))$$

Given finite automata $M_1, M_2$ such that

$$L_1 = L(M_1) \quad \text{and} \quad L_2 = L(M_2)$$

We construct $M$ such that $L(M) = L_1 \cap L_2$ as follows

**1.** Transform $M_1$, $M_2$ into equivalent DFA automata $N_1, N_2$

**2.** Construct $\overline{N_1}$, $\overline{N_2}$ and then $N = \overline{N_1} \cup \overline{N_2}$

**3.** Transform NDF automaton $N$ into equivalent DFA automaton $N'$

**4.** $M = \overline{N'}$ is the required finite automata

This is an indirect Construction

**Homework**: describe the direct construction

**CLOSURE THEOREM**

The class of languages accepted by **Finite Automata** FA) is **closed** under the following operations

1. union **proved**
2. concatenation **proved**
3. Kleene's Star **proved**
4. complementation **proved**
5. intersection **proved**

**Observe** that we used the term **Finite Automata** (FA) so in the

proof we can choose a DFA or NDFA, as we have already proved their **equivelency**

**Direct Construction**

**Case 1** deterministic

Given **deterministic** automata $M_1, \ M_2$ such that

$$M_1 = (K_1, \ \Sigma_1, \ \delta_1, \ s_1, \ F_1), \qquad M_2 = (K_2, \ \Sigma_2, \ \delta_2, \ s_2, \ F_2)$$

We construct $M = M_1 \cap M_2$ such that $L(M) = L(M_1) \cap L(M_2)$ as follows

$$M = (K, \ \Sigma, \delta, \ s, \ F)$$

where . $\Sigma = \Sigma_1 \cup \Sigma_2$

$$K = K_1 \times K_2, \qquad s = (s_1, s_2), \qquad F = F_1 \times F_2$$

$$\delta((q_1, q_2), \ \sigma) = (\delta_1(q_1, \ \sigma), \ \delta_2(q_2, \ \sigma))$$

# Intersection Direct Construction

**Proof** of correctness of the construction

$w \in L(M)$ if and only if

$((s_1, s_2), w) \vdash_M{}^* ((f_1, f_2), e))$ and $f_1 \in F_1, f_2 \in F_2$

if and only if

$(s_1, w) \vdash_{M_1}{}^* (f_1, e)$ for $f_1 \in F_1$ and

$(s_2, w) \vdash_{M_2}{}^* (f_2, e)$ for $f_2 \in F_2$

if and only if

$w \in L(M_1)$ and $w \in L(M_2)$

if and only if

$w \in L(M_1) \cap L(M_2)$

# Intersection Direct Construction

**Direct Construction**

**Case 2** nondeterministic

Given **nondeterministic** automata $M_1, \ M_2$ such that

$$M_1 = (K_1, \ \Sigma_1, \ \Delta_1, \ s_1, \ F_1), \qquad M_2 = (K_2, \ \Sigma_2, \ \Delta_2, \ s_2, \ F_2)$$

We construct $M = M_1 \cap M_2$ such that $L(M) = L(M_1) \cap L(M_2)$ as follows

$$M = (K, \ \Sigma, \Delta, \ s, \ F)$$

where $\Sigma = \Sigma_1 \cup \Sigma_2$

$$K = K_1 \times K_2, \quad s = (s_1, s_2), \quad F = F_1 \times F_2$$

and $\Delta$ is defined as follows

# Intersection Direct Construction

$\Delta$ is defined as follows

$$\Delta = \Delta' \cup \Delta'' \cup \Delta'''$$

$\Delta' = \{((q_1, q_2), \sigma, (p_1, p_2)) : \ (q_1, \sigma, p_1) \in \Delta_1$ and $(q_2, \sigma, p_2) \in \Delta_2, \ \ \sigma \in \Sigma\}$

$\Delta'' = \{((q_1, q_2), \sigma, (p_1, p_2)) : \ \sigma = e, \ \ (q_1, \ e, \ p_1) \in \Delta_1$ and $q_2 = p_1\}$

$\Delta'' = \{((q_1, q_2), \sigma, (p_1, p_2)) : \ \sigma = e, \ (q_2, e, p_2) \in \Delta_2$ and $q_1 = p_1\}$

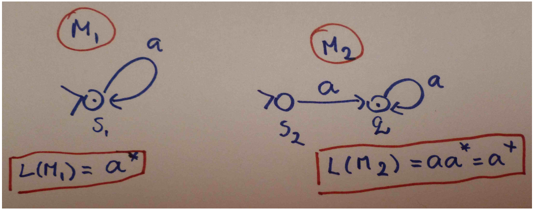
**Observe** that if $M_1, M_2$ have each at most n states, our direct construction of produces $M = M_1 \cap M_2$ with at most $n^2$ states.

The **indirect** construction from the proof of the theorem might generate $M$ with up to $2^{2^{n+1}+1}$ states

## Direct Construction Example

**Example**

Let $M_1$, $M_2$ be given by the following **diagrams**




**Observe** that $L(M_1) \cap L(M_2) = a^* \cap a^+ = a^+$

# Direct Construction Example

Formally $M_1, \ M_2$ are defined as follows

$$M_1 = (\{s_1\}, \ \{a\}, \ \delta_1, \ s_1, \ \{s_1\}), \ \ M_2 = (\{s_2, q\}, \ \{a\}, \ \delta_2, \ s_2, \ \{q\})$$

for $\delta_1(s_1, a) = s_1$ and $\delta_2(s_2, a) = q, \ \delta_2(q, a) = q$

By the deterministic case **definition** we have that
$M = M_1 \cap M_2$ is

$$M = (K, \ \Sigma, \delta, \ s, \ F)$$

for $\Sigma = \{a\}$

$$K = K_1 \times K_2 = \{s_1\} \times \{s_2, q\} = \{(s_1, s_2), \ (s_1, g) \ \}$$

$$s = (s_1, s_2), \quad F = \{s_1\} \times \{q\} = \{(s_1, q)\}$$

# Direct Construction Example

By definition

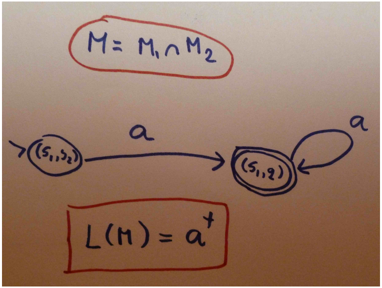
$$\delta((q_1, q_2),\ \sigma) = (\delta_1(q_1,\ \sigma),\ \delta_2(q_2,\ \sigma))$$

In our case we have

$$\delta((s_1, s_2),\ a) = (\delta_1(s_1,\ a),\ \delta_2(s_2,\ a)) = (s_1, q),$$

$$\delta((s_1, q),\ a) = (\delta_1(s_1,\ a),\ \delta_2(q,\ a)) = (s_1, q)$$

The **diagram** of $M = M_1 \cap M_2$ is



# Main Theorem

Now our goal is to prove a theorem that established the relationship between languages and finite automata

This is the most important Theorem of this section so we call it a Main Theorem

**Main Theorem**

A language L is regular

if and only if

L is accepted by a finite automata

# Main Theorem

The **Main Theorem** consists of the following two parts

**Theorem 1**

For any a regular language L

there is a e finite automata M, such that $L = L(M)$

**Theorem 2**

For any a finite automata M, the language L(M) is regular

# Main Theorem

**Definition**

A language $L \subseteq \Sigma^*$ is regular if and only if there is a regular expression $r \in \mathcal{R}$ that represents L, i.e. such that

$$L = \mathcal{L}(r)$$

**Reminder**: the function $\mathcal{L} : \mathcal{R} \longrightarrow 2^{\Sigma^*}$ is defined recursively as follows

**1.** $\mathcal{L}(\emptyset) = \emptyset$, $\mathcal{L}(\sigma) = \{\sigma\}$ for all $\sigma \in \Sigma$

**2.** If $\alpha, \beta \in \mathcal{R}$, then

$$\mathcal{L}(\alpha\beta) = \mathcal{L}(\alpha) \circ \mathcal{L}(\beta) \quad \text{concatenation}$$

$$\mathcal{L}(\alpha \cup \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta) \quad \text{union}$$

$$\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^* \quad \text{Kleene's Star}$$

## Regular Expressions Definition

**Reminder**

We define a $\mathcal{R}$ of **regular expressions** over an alphabet $\Sigma$ as follows

$\mathcal{R} \subseteq (\Sigma \cup \{(,\ ),\ \emptyset,\ \cup,\ *\})^*$ and $\mathcal{R}$ is the smallest set such that

**1.** $\emptyset \in \mathcal{R}$ and $\Sigma \subseteq \mathcal{R}$, i.e. we have that

$$\emptyset \in \mathcal{R} \text{ and } \forall_{\sigma \in \Sigma} (\sigma \in \mathcal{R})$$

**2.** If $\alpha, \beta \in \mathcal{R}$, then

$(\alpha\beta) \in \mathcal{R}$ concatenation

$(\alpha \cup \beta) \in \mathcal{R}$ union

$\alpha^* \in \mathcal{R}$ Kleene's Star

# Proof of Main Theorem Part 1

Now we are going to **prove** the first part of the Main Theorem, i.e.

**Theorem 1**

For any a regular language L

there is a finite automata M, such that $L = L(M)$

**Proof**

By definition of regular language, L is regular if and only if

there is a regular expression $r \in \mathcal{R}$ that represents L, what

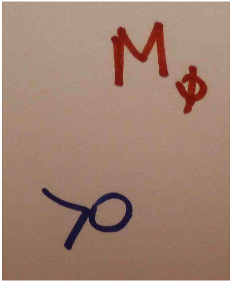
we write in **shorthand** notation as L = r

Given a regular language, L, we **construct** a finite automaton M such that L(M) = L recursively following the definition of the set $\mathcal{R}$ of **regular expressions** as follows

## Proof Theorem 1

**1.** $r = \emptyset$, i.e. the language is $L = \emptyset$

**Diagram** of M, such that $L(M) = \emptyset$ is

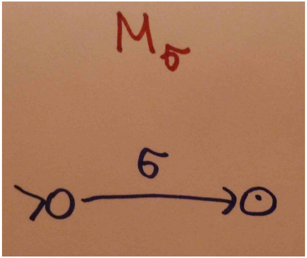



We denote M as $M = M_\emptyset$

## Proof Theorem 1

**2.** $r = \sigma$, for any $\sigma \in \Sigma$ i.e. the language is $L = \sigma$

**Diagram** of M, such that $L(M) = \emptyset$ is




We denote M as $M = M_\sigma$

# Proof Theorem 1

**3.** $r \neq \emptyset, \quad r \neq \sigma$

By the recursive definition, we have that $L = r$ where

$$r = \alpha \cup \beta, \quad r = \alpha \circ \beta, \quad r = \alpha^*$$

for any $\alpha, \beta \in \mathcal{R}$

We construct as in the proof of the **Closure Theorem** the automata

$$M_r = M_\alpha \cup M_\beta, \quad M_r = M_\alpha \circ M_\beta, \quad M_r = (M_r)^*$$

respectively and it ends the proof

# Example

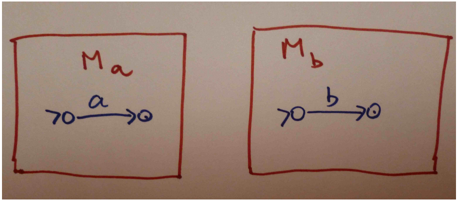
Use construction defined in the proof of **Theorem 1** to construct an automaton M such that

$$L(M) = (ab \cup aab)^*$$

We construct M in the following stages

**Stage 1**

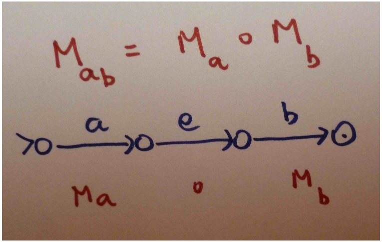
For $a, b \in \Sigma$ we construct $M_a$ and $M_b$



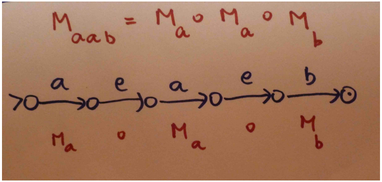
# Example

**Stage 2**

For $ab$, $aab$ we use $M_a$ and $M_b$ and **concatenation** construction to construct $M_{ab}$



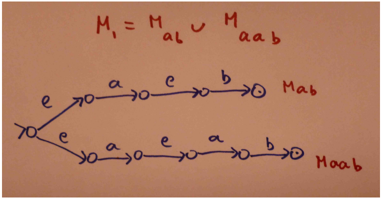
and $M_{aab}$



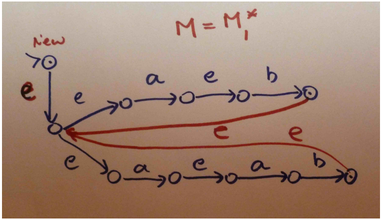
## Example

**Stage 3**

We use **union** construction to construct $M_1 = M_{ab} \cup M_{aab}$




**Stage 4** We use Kleene's **star** construction to construct $M = {M_1}^*$



## Exercise

Use construction defined in the proof of **Theorem 1** to construct an automaton M such that

$$L(M) = (a^* \cup abc \cup a^*b)^*$$

We construct (draw diagrams) M in the following stages

**Stage 1**

Construct $M_a, \quad M_b, \quad M_c$

**Stage 2**

Construct $M_1 = M_{abc}$

**Stage 3**

Construct $M_2 = {M_a}^*$

**Stage 4**

Construct $M_3 = {M_a}^* M_b$

**Stage 5**

Construct $M_4 = M_1 \cup M_2 \cup M_3$

**Stage 6**

Construct $M = {M_4}^*$

# Main Theorem Part 2

**Theorem 2**

For any a finite automaton M there is a regular expression $r \in \mathcal{R}$, such that

$$L(M) = r$$

**Proof**

The proof is **constructive**; given M we will give an algorithm how to recursively generate the regular expression r , such that $L(M) = r$

We assume that M is nondeterministic

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

We use the BOOK definition, i.e.

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

# Proof of Theorem 2

We put states of M into a one- to - one sequence

$$K: \quad s = q_1,\ q_2,\ \dots\ q_n \text{ for } n \geq 1$$

We build r using the following expressions

$$R(i,\ j,\ k) \quad \text{for } i, j = 1,\ 2,\ \dots\ n, \quad k = 0,\ 1,\ 2,\ \dots\ n$$

$$R(i,\ j,\ k) = \{w \in \Sigma^*; \quad (q_i,\ w) \vdash_{M,k}{}^*(q_j, w')\}$$

R(i, j, k) is the set of all words "spelled" by all PATHS from $q_i$ to $q_j$ in such way that we **do not pass** through an intermediate state numbered k+1 or greater

**Observe** that $\neg(m \geq k+1) \equiv m \leq k$ so we get the following

## Proof of Theorem 2

We say that a PATH has a RANK k when

$$(q_i,\ w) \vdash_{M,k}{}^{*} (q_j, w')$$

I.e. when M can pass ONLY through states numbered $m \leq k$ while going from $q_i$ to $q_j$

RANK 0 **case** $k = 0$

$$R(i,\ j,\ 0) = \{w \in \Sigma^{*};\quad (q_i,\ w) \vdash_{M,0}{}^{*} (q_j, w')\}$$

This means; M "goes" from $q_i$ to $q_j$ only through states numbered $m \leq 0$

There is **no** such states as $K = \{q_1,\ q_2,\ \ldots\ \ q_n\}$

# Proof of Theorem 2

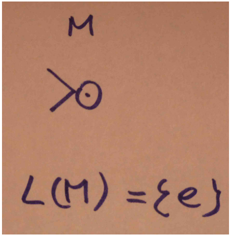
Hence $R(i,\ j,\ 0)$ means that M "goes" from $q_i$ to $q_j$ DIRECTLY, i.e. that

$$R(i,\ j,\ 0) = \{w \in \Sigma^*;\quad (q_i,\ w) \vdash_M{}^* (q_j, w')\}$$

**Reminder**: we use the BOOK definition so

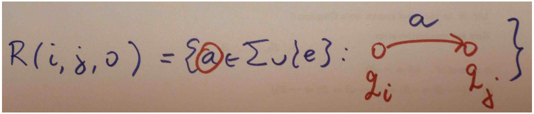
$$R(i,\ j,\ 0) = \begin{cases} a \in \Sigma \cup \{e\} & \text{if } i \neq j \text{ and } (q_i, a, q_j) \in \Delta \\ \{e\} \cup a \in \Sigma \cup \{e\} & \text{if } i = j \text{ and } (q_i, a, q_j) \in \Delta \end{cases}$$

**Observe** that we need $\{e\}$ in the second equation to include the following special case

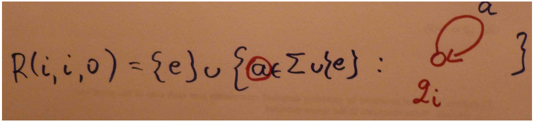


# Proof of Theorem 2

We read $R(i, j, 0)$ from the **diagram** of M as follows




and



# Proof of Theorem 2

RANK n **case** $k = n$

$$R(i,\ j,\ n) = \{w \in \Sigma^*;\quad (q_{\ i},\ w) \vdash_{M,n}{}^{*} (q_{\ j}, w')\}$$

This means; M "goes" from $q_i$ to $q_j$ through states numbered $m \leq n$

It means that M "goes" all states as $|K| = n$

It means that M will read any $w \in \Sigma$ and hence

$$R(i,\ j,\ n) = \{w \in \Sigma^*;\quad (q_i,\ w) \vdash_{M}{}^{*} (q_j, e)\}$$

**Observe** that

$$w \in L(M) \quad \text{iff} \quad w \in R(1,\ j,\ n) \quad \text{and} \quad q_j \in F$$

## Proof of Theorem 2

By definition of the $L(M)$ we get

$$L(M) = \bigcup \{R(1, j, n) : \ q_j \in F\}$$

**Fact**

All sets $R(i, j, k)$ are regular and hence L( M) is also regular

**Proof** by induction on k

**Base case**: k =0

All sets R(i, j, 0) are FINITE, hence are regular

## Proof of Theorem 2

**Inductive Step**

The **recursive formula** for $R(i,\ j,\ k)$ is

$$R(i,j,k) = R(i,j,k-1) \cup R(i,k,k-1)R(k,k,k-1)^{*}R(k,j,k-1)$$

where n is the number of states of M and
$k = 0, \ldots, n, \ \ i,j = 1, \ldots, n$

By Inductive assumption, all sets
$R(i,j,k-1),\ R(i,k,k-1),\ R(k,k,k-1),\ R(k,j,k-1)$ are regular and by the **Closure Theorem** so is the set $R(i,j,k)$

This **ends** the proof of **Theorem 2**

**Observe** that the recursive formula for $R(i,\ j,\ k)$ computes r such that L(M) = r

## Example

**Example**

For the automaton M such that

$$M = (\{q_1, q_2, q_3\},\ \{a, b\},\ s = q_1,$$

$$\Delta = \{(q_1, b, q_2), (q_1, a, q_3), (q_2, a, q_1), (q_2, b, q_1),$$

$$(q_3, a, q_1), (q_3, b, q_1)\},\ F = \{q_1\})$$

**Evaluate 4 steps**, in which you must include at least one $R(i, j, 0)$, in the construction of regular expression that defines $L(M)$

# Example

**Reminder**

$$L(M) = \bigcup \{R(1,\ j,\ n) :\ \ q_j \in F\}$$

$$R(i,j,k) = R(i,j,k-1) \cup R(i,k,k-1)R(k,k,k-1)^* R(k,j,k-1)$$

$$R(i,\ j,\ 0) = \begin{cases} a \in \Sigma \cup \{e\} & \text{if } i \neq j \text{ and } (q_i, a, q_j) \in \Delta \\ \{e\} \cup a \in \Sigma \cup \{e\} & \text{if } i = j \text{ and } (q_i, a, q_j) \in \Delta \end{cases}$$

## Example Solution

**Solution**

**Step 1** $L(M) = R(1,1,3)$

**Step 2**

$R(1,1,3) = R(1,1,2) \cup R(1,3,2)R(3,3,2)^* R(3,1,2)$

**Step 3**

$R(1,1,2) = R(1,1,1) \cup R(1,2,1)R(2,2,1)^* R(2,1,1)$

**Step 4**

$R(1,1,1) = R(1,1,0) \cup R(1,1,0)R(1,1,0)^* R(1,1,0)$ and

$R(1,1,0) = \{e\} \cup \emptyset = \{e\}$, so we get

$R(1,1,1) = \{e\} \cup \{e\}\{e\}^*\{e\} = \{e\}$

# Generalized Automata

# Generalized Automaton

**Definition**

We define now a **Generalized Automaton** GM as the following generalization of of a nondeterministic automaton $M = (K, \Sigma, \Delta, s, F)$ as follows

$$GM = (K_G, \Sigma_G, \Delta_G, s_G, F_G)$$

**1.** GM has a single final state, i,e. $F_G = \{f\}$

**2.** $\Sigma_G = \Sigma \cup \mathcal{R}_0$ where $\mathcal{R}_0$ is a FINITE subset of the set $\mathcal{R}$ of **regular expressions** over $\Sigma$

**3.** Transitions of GM may be labeled not only by symbols in $\Sigma \cup \{e\}$ but also by **regular expressions** $r \in \mathcal{R}$, i.e. $\Delta_G$ is a FINITE set such that

$$\Delta_G \subseteq K \times (\Sigma \cup \{e\} \cup \mathcal{R}) \times K$$

**4.** There is no transition going into the initial state $s$ nor out of the final state $f$

if $(q, u, p) \in \Delta_G$, then $q \neq f$, $p \neq s$

## Generalized Automata

Given a nondeterministic automaton

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

We present now a new method of construction of a regular expression $r \in \mathcal{R}$ that defines $L(M)$, i.e. such that $L(M) = r$ by the use of the notion of of **Generalized Automaton**

The method consists of a construction of a sequence of generalized automata that are all equivalent to M

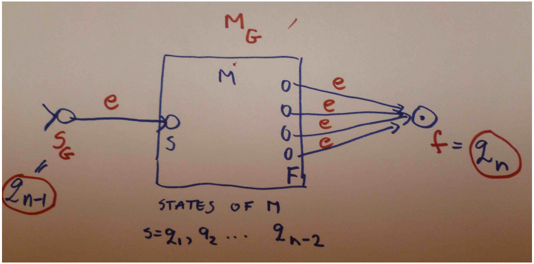
# Construction

**Steps** of construction are as follows

**Step 1**

We **extend** $M$ to a generalized automaton $M_G$, such that $L(M) = L(M_G)$ as depicted on the diagram below

**Diagram** of $M_G$



## $M_G$ Definition

**Definition** of $M_G$

We re-name states of M as $s = q_1, q_2, \ldots, q_{n-2}$ for appropriate n and make the initial state $s = q_1$ and all final states of M the internal non-final states of $G_M$

We ADD TWO states: initial and one final, which me name $q_{n-1}$, $q_n$, respectively, i.e. we put

$$s_G = q_{n-1} \quad \text{and} \quad f = q_n$$

We take

$$\Delta_G = \Delta \cup \{(q_{n-1}, e, s)\} \cup \{(q, e, q_n) : \; q \in F\}$$

**Obviously** $L(M) = L(M_G)$, and so $M \approx M_G$

## States of $G_M$ Elimination

We construct now a sequence $GM1, GM2, \ldots, GM(n-2)$ such that

$$M \approx M_G \approx GM1 \approx \cdots \approx GM(n-2)$$

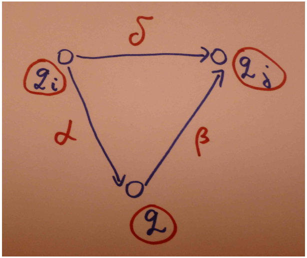
where $GM(n-2)$ has only **two states** $q_{n-1}$ and $q_n$ and only **one transition** $(q_{n-1}, r, q_n)$ for $r \in \mathcal{R}$, such that

$$L(M) = r$$

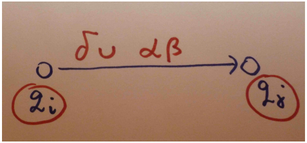
We construct the sequence $GM1, GM2, \ldots, GM(n-2)$ by eliminating states of M one by one following rules given by the following diagrams

**Case 1** of state **elimination**

Given a fragment of *GM* **diagram**




we **transform** it into

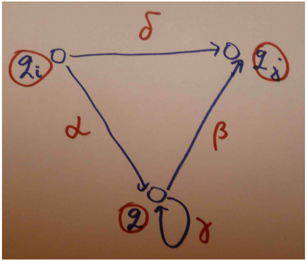



The state $q \in K$ has been **eliminated** preserving the language of *GM* and we constructed $GM' \approx GM$

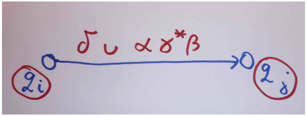
# States of $G_M$ Elimination

**Case 2** of state **elimination**

Given a fragment of *GM* **diagram**




we **transform** it into




The state $q \in K$ has been **eliminated** preserving the language of *GM* and we constructed $GM' \approx GM$

**Example 1**

Use the Generalized Automata Construction and States of $G_M$ Elimination procedure to evaluate $r \in \mathcal{R}$, such that

$$\mathcal{L}(r) = L(M)$$

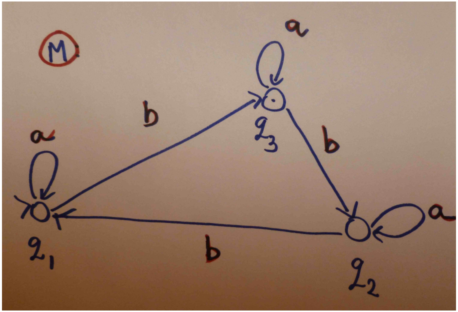
, where M is an automata that accepts the language

$$L = \{w \in \{a,b\}^* : \quad w \text{ has } \; 3k+1 \; b's, \; \text{for some } \; k \in N\}$$

This is the Book example, page 80

# Example 1

The **Diagram** of M is




**Step 1**

We extend M with $K = \{q_1, q_2, q_3\}$ to a generalized $M_G$ by adding two states

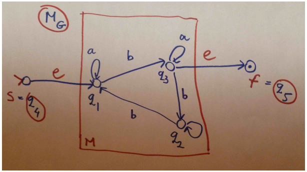
$$s_G = q_4 \quad \text{and} \quad f = q_5$$

We take

$$\Delta_G = \Delta \cup \{(q_4, e, q_1)\} \cup \{(q_3, e, q_5)\}$$

# Example 1

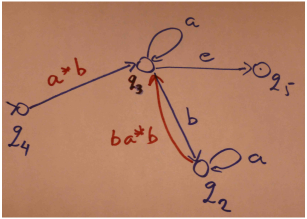
The **Diagram** of $M_G$ is




**Step 2**

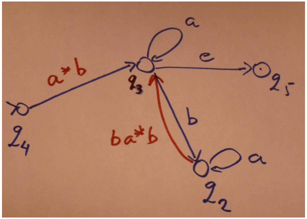
We construct $GM1 \approx M_G \approx M$ by **elimination** of $q_1$

The **Diagram** of $GM1$ is



# Example 1

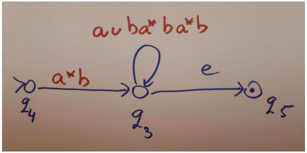
The **Diagram** of $GM1$ is




**Step 3**

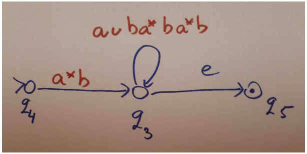
We construct $GM2 \approx GM1$ by **elimination** of $q_2$

The **Diagram** of $GM2$ is



# Example 1

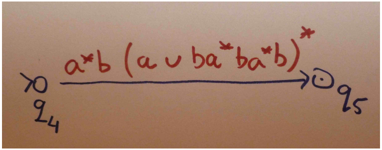
The **Diagram** of *GM2* is




**Step 4**

We construct $GM3 \approx GM2$ by **elimination** of $q_3$

The **Diagram** of *GM2* is




$$L(GM3) = a^*b(a \cup ba^*ba^*b)^* = L(M)$$

**Example 2**

Given the automaton

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

where

$$K = \{q_1, q_2, q_3\}, \quad \Sigma = \{a, b\}, \quad s = q_1, \quad F = \{q_1\}$$

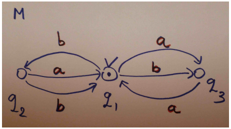
$$\Delta = \{(q_1, b, q_2),\ (q_1, a, q_3),\ (q_2, a, q_1),$$
$$(q_2, b, q_1),\ (q_3, a, q_1),\ (q_3, b, q_1)$$

Use the Generalized Automata Construction and States of $G_M$ Elimination procedure to evaluate $r \in \mathcal{R}$, such that

$$\mathcal{L}(r) = L(M)$$

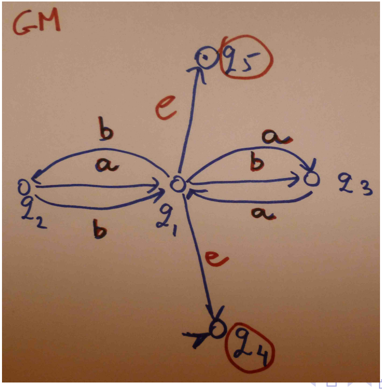
# Example 2

The **diagram** of M is




**Step 1**

The **diagram** of $M_G \approx M$ is



## Example 2

**Step 1**

The components of $M_G \approx M$ are

$$M_G = (K = \{q_1, q_2, q_3, q_4, q_5\}, \quad \Sigma = \{a, b\}, \quad s_G = q_4,$$

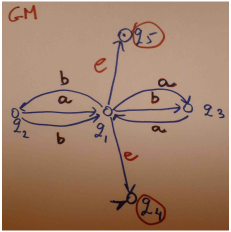
$$\Delta_G = \{(q_1, b, q_2),\ (q_1, a, q_3),\ (q_2, a, q_1),$$

$$(q_2, b, q_1), (q_3, a, q_1), (q_3, b, q_1),\ (q_4, e, q_1),$$

$$(q_1, e, q_5)\}, \qquad F = \{q_5\})$$

# Example 2

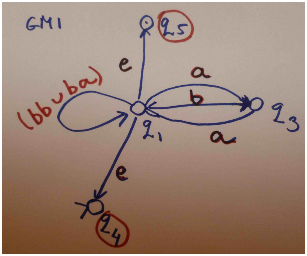
The **Diagram** of $M_G$ is




**Step 2**

We construct $GM1 \approx M_G \approx M$ by **elimination** of $q_2$

The **Diagram** of $GM1$ is



## Example 2

**Step 2**

The components of $GM1 \approx M_G \approx M$ are

$$GM1 = (K = \{q_1, q_3, q_4, q_5\}, \quad \Sigma = \{a, b\}, \quad s_G = q_4$$

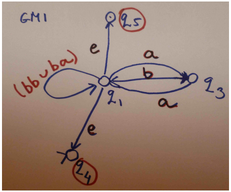
$$\Delta_G = \{(q_1, a, q_3), \; (q_1, (bb \cup ba), q_1),$$

$$(q_3, a, q_1), \; (q_3, b, q_1), \; (q_4, e, q_1),$$

$$(q_1, e, q_5)\}, \quad F = \{q_5\})$$

# Example 2

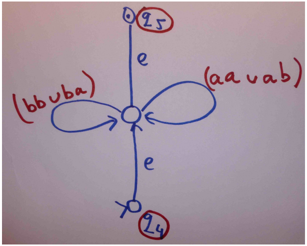
The **Diagram** of *GM1* is




**Step 3**

We construct $GM2 \approx GM1$ by **elimination** of $q_3$

The **Diagram** of *GM2* is



## Example 2

**Step 3**

The components of $GM2 \approx GM1 \approx M_G \approx M$ are

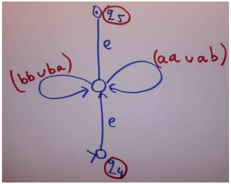
$$GM2 = (K = \{q_1,\ q_4,\ q_5\}, \quad \Sigma = \{a, b\}, \quad s_G = q_4$$

$$\Delta_G = \{(q_1, (bb \cup ba), q_1), \ (q_1, (aa \cup ab), q_1),$$

$$(q_4, e, q_1), (q_1, e, q_5)\}, \quad F = \{q_5\})$$

# Example 2

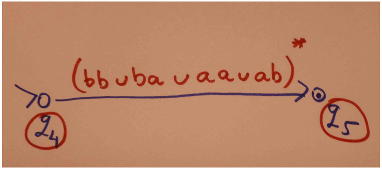
The **Diagram** of $GM2$ is




**Step 4**

We construct $GM3 \approx GM2$ by **elimination** of $q_1$

The **Diagram** of $GM3$ is

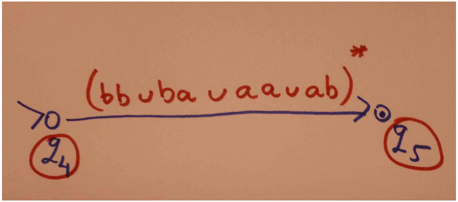


## Example 2

We have constructed

$$GM3 \approx GM2 \approx GM1 \approx M_G \approx M$$

The **Diagram** of $GM3$ is




Hence the language

$$L(GM3) = (bb \cup ba \cup aa \cup ab)^* = ((a \cup b)(a \cup b))^* = L(M)$$