cse303
ELEMENTS OF THE THEORY OF COMPUTATION

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LECTURE 6

CHAPTER 2
FINITE AUTOMATA

1. Deterministic Finite Automata DFA
2. Nondeterministic Finite Automata NFA
3. Finite Automata and Regular Expressions
4. Languages that are Not Regular
5. State Minimization

CHAPTER 2
PART 2: Nondeterministic Finite Automata NDFA

NDFA: Nondeterministic Finite Automata

Now we add a new powerful feature to the **finite automata**

This feature is called **nondeterminism**

Nondeterminism is essentially the ability to change states in a way that is only **partially determined** by the current state and input symbol, or a string of symbols, empty string included

The automaton, as it reads the input string, may choose at each step to go to any of its states

The choice is not determined by anything in our model , and therefore it is said to be **nondeterministic**

At each step there is always a finite number of choices, hence it is still a **finite automaton**

NDFA - Mathematical Model

Class Definition

A Nondeterministic Finite Automata is a quintuple

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

where

K is a finite set of **states**

Σ as an **alphabet**

$s \in K$ is the **initial state**

$F \subseteq K$ is the set of **final states**

Δ is a **finite set** and

$$\Delta \subseteq K \times \Sigma^* \times K$$

Δ is called the **transition relation**

We usually use different symbols for $K,\ \Sigma$, i.e. we have that $K \cap \Sigma = \emptyset$

NDFA Definition

Class Definition revisited

A Nondeterministic Finite Automata is a quintuple

$$M = (K, \Sigma, \Delta, s, F)$$

where

K is a finite set of **states**

$K \neq \emptyset$ because $s \in K$

Σ as an **alphabet**

Σ can be $\emptyset$ - case to consider

$s \in K$ is the **initial state**

$F \subseteq K$ is the set of **final states**

F can be $\emptyset$ - case to consider

Δ is a **finite set** and $\Delta \subseteq K \times \Sigma^* \times K$

Δ is called the **transition relation**

Δ can be $\emptyset$ - case to consider

Some Remarks

R1 We **must** say that Δ is a **finite** set because the set $K \times \Sigma^* \times K$ is countably infinite, i.e. $|K \times \Sigma^* \times K| = \aleph_0$) and we want to have a **finite automata** and we defined it as

$$\Delta \subseteq K \times \Sigma^* \times K$$

R2 The DFA **transition function** $\delta : K \times \Sigma \longrightarrow K$ is (as any function!) a **relation**

$$\delta \subseteq K \times \Sigma \times K$$

R3 The **set** δ is always **finite** as the **set** $K \times \Sigma \times K$ is **finite**

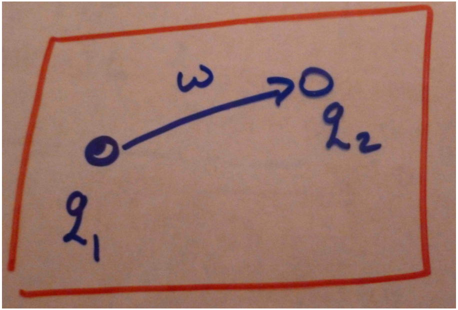
R4 The DFA **transition function** δ is a particular case of the NDFA **transition relation** Δ, hence similarity of notation

NDFA Diagrams

We extend the notion of the **state diagram** to the case of the NDFA in natural was as follows

$(q_1, w, q_2) \in \Delta$ means that M in a state q_1 reads the word $w \in \Sigma^*$ and goes to the state q_2

Picture

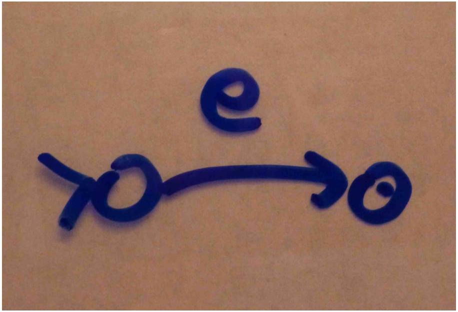


Remember that in particular w = e

Examples

Example 1

Let M be given by a diagram

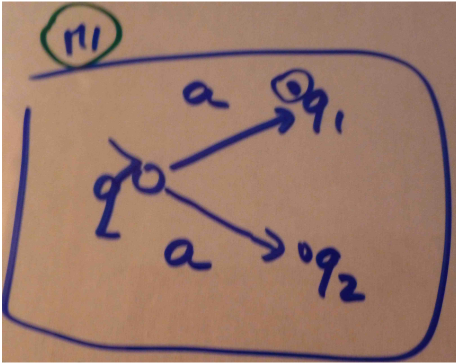


By definition M **is not** a deterministic DFA as it reads $e \in \Sigma^*$

$$L(M) = \{e\}$$

Example 2

Let M1 be given by a diagram

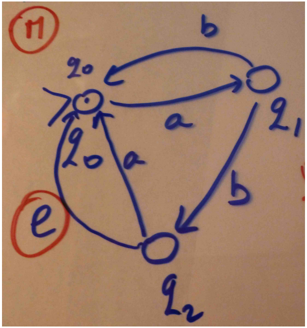


Observe that M1 **is not** a deterministic DFA as $(q, a, q_1) \in \Delta$ and $(q, a, q_2) \in \Delta$ what proves that Δ is **not a function**

$$L(M1) = \{a\}$$

Example 3

Let M be given by a diagram



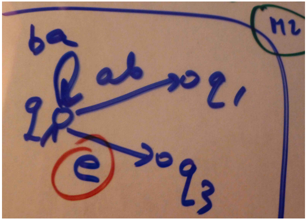
M **is not** a deterministic DFA as $(q_2, e, q_0) \in \Delta$ and this is not admitted in DFA

$\Delta = \{(q_0, a, q_1),\ (q_1, b, q_0),\ (q_1, b, q_2),\ (q_2, a, q_0), (q_2, e, q_0)\}$

Examples

Example 4

Let M be given by a diagram



M **is not** a deterministic DFA as $(q, ab, q_1) \in \Delta$ and this is not admitted in DFA

$\Delta = \{(q, ba, q), (q, ab, q_1), \ (q, e, q_3)\}$ and $F = \emptyset$

$$L(M1) = \emptyset$$

NDFA - Book Definition

Book Definition

A **Nondeterministic Finite Automata** is a quintuple

$$M = (K, \Sigma, \Delta, s, F)$$

where

K is a finite set of **states**

Σ as an **alphabet**

$s \in K$ is the **initial state**

$F \subseteq K$ is the set of **final states**

Δ , the **transition relation** is defined as

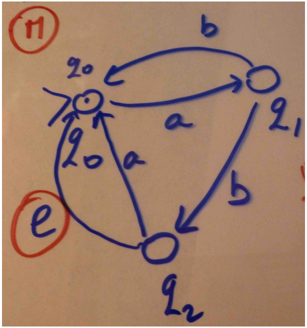
$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

Observe that Δ is finite set as both K and $\Sigma \cup \{e\}$ are finite sets

Book Definition Example

Example

Let M be automaton from **Example 3** given by a diagram



M follows the **Book Definition** as

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

Equivalence of Definitions

The Class and the Book definitions are equivalent

1. We get the **Book Definition** as a particular case of the **Class Definition** as

$$\Sigma \cup \{e\} \subseteq \Sigma^*$$

2. We will show later a general method how to transform any automaton defined by the **Class Definition** into an equivalent automaton defined by the **Book Definition**

When solving problems you can use any of these definitions

Configuration and Transition Relation

Given a NDFA automaton

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

We define as we did in the case of DFA the notions of a configuration, and a transition relation

Definition

A **configuration** in a NDFA is any tuple

$$(q, w) \in K \times \Sigma^*$$

Configuration and Transition Relation

Definition

A **transition relation** in $M = (K, \Sigma, \Delta, s, F)$

defined by the **Class Definition** is a binary relation

$$\vdash_M \subseteq (K \times \Sigma^*) \times (K \times \Sigma^*)$$

such that $q, q' \in K$, $u, w \in \Sigma^*$

$$(q, uw) \vdash_M (q', w)$$

if and only if

$$(q, u, q') \in \Delta$$

For M defined by the **Book Definition** definition of the Transition Relation is the same but for the fact that

$$u \in \Sigma \cup \{e\}$$

Language Accepted by M

We define, as in the case of the deterministic DFA , the language accepted by the **nondeterministic** M as follows

Definition

$$L(M) = \{w \in \Sigma^* : (s, w) \vdash_M{}^* (q, e) \text{ for } q \in F\}$$

where $\vdash_M{}^*$ is the reflexive, transitive closure of $\vdash_M$

Equivalency of Automata

We define now formally an equivalency of automata as follows

Definition

For any two automata M_1, M_2 (deterministic or nondeterministic)

$$M_1 \approx M_2 \quad \text{if and only if} \quad L(M_1) = L(M_2)$$

Now we are going to formulate and prove the main theorem of this part of the Chapter 2, informally stated as

Equivalency Statement

The notions of a deterministic and a non-dederteministic automata are **equivalent**

Equivalency of Automata Theorems

The **Equivalency Statement** consists of two **Equivalency Theorems**

Equivalency Theorem 1

For any **DFA** M, there is is a **NDFA** M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

Equivalency Theorem 2

For any **NDFA** M, there is is a **DFA** M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

Equivalency of Automata Theorems

Equivalency Theorem 1

For any **DFA** M, there is is a **NDFA** M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

Proof

Any **DFA** M is a particular case of a **DFA** M' because any function δ is a relation

Moreover δ and its a particular case of the relation Δ as $\Sigma \subseteq \Sigma \cup \{e\}$ (for the Book Definition) and $\Sigma \subseteq \Sigma^*$ (for the Class Definition)

This ends the **proof**

Equivalency Theorem 2

For any **NDFA** M, there is is a **DFA** M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

Proof

The proof is far from trivial. It is a constructive proof;

We will describe, given a **NDFA** M, a general method of construction step by step of an **DFA** M' that accepts the came language as M

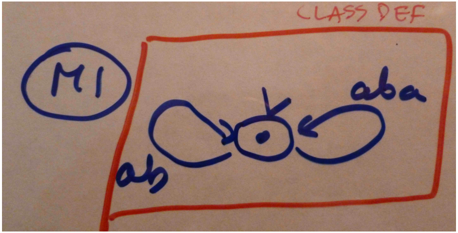
Before we define the **poof** construction we discuss some examples and some general automata properties

EXAMPLES and QUESTIONS

Examples

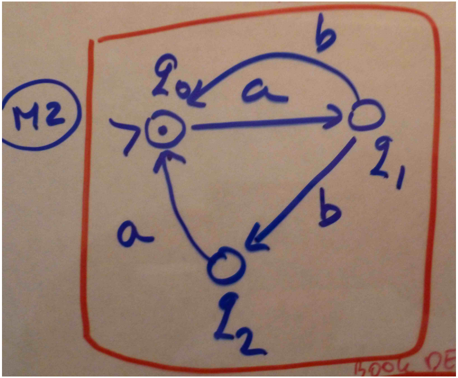
Example 1

Here is a **diagram** of NDFA M1 - Class Definition



$$L(M1) = (ab \cup aba)^*$$

Example 2
Here is a **diagram** of NDFA M2 - Book Definition



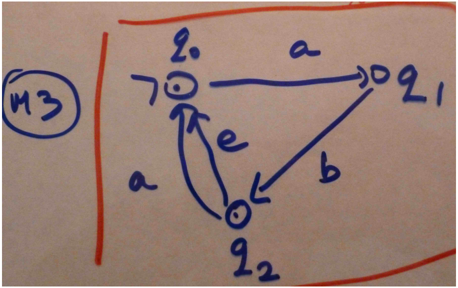
Observe that M2 **is not deterministic** (even if we add "plus trap states) because Δ is **not a function** as $(q_1, b, q_0) \in \Delta$ and $(q_1, b, q_2) \in \Delta$

$$L(M2) = (ab \cup aba)^*$$

Examples

Example 3

Here is a **diagram** of NDFA M3 - Book Definition



Observe that M2 **is not deterministic** $(q_1, e, q_0) \in \Delta$

$$L(M3) = (ab \cup aba)^*$$

Question 1

All automata in **Examples 1-3** accept the same language, hence by definition, they are **equivalent nondeterministic** automata, i.e.

$$M1 \approx M2 \approx M3$$

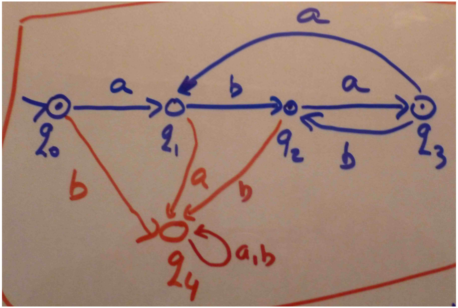
Question 1

Construct a **deterministic** automaton M4 such that

$$M1 \approx M2 \approx M3 \approx M4$$

Question1 Solution

Here is a **diagram** of **deterministic** DFA M4



Observe that q_4 is a **trap state**

$$L(M4) = (ab \cup aba)^*$$

Question 2

Given an alphabet

$$\Sigma = \{a_1,\ a_2, \ldots, a_n\} \quad \text{for} \quad n \geq 2$$

Question 2

Construct a **nondeterministic** automaton M such that

$$L = \{w \in \Sigma^* :\ \text{at least one letter from } \Sigma \text{ is missing in } w\ \}$$

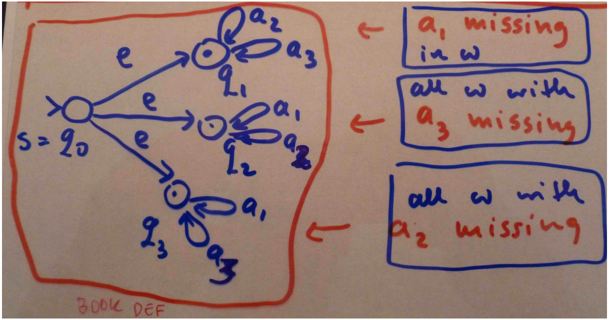
Take $n = 4$, i.e. $\Sigma = \{a_1,\ a_2,\ a_3,\ a_4\}$

Some words in L are:

$$e \in L,\ \ a_1 \in L,\ a_1 a_2 a_3 \in L,\ a_1 a_2 a_2 a_3 a_3 \in L\ a_1 a_4 a_1 a_2 \in L, \ldots$$

Question 2 Solution

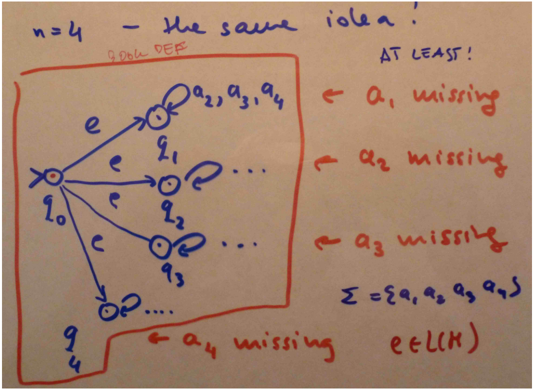
Here is **solution** for $n = 3$, i.e. $\Sigma = \{a_1,\ a_2,\ a_3\}$



Write a solution for $n = 4$

Question 2 Solution

Here is the **solution** for $n = 4$, i.e. $\Sigma = \{a_1, a_2, a_3, a_4\}$



Write a **general** form of solution for $n \geq 2$

Question 2 Solution

General case

$M = (K, \ \Sigma, \ \Delta, \ s, \ F)$ for $\Sigma = \{a_1, \ a_2, \ldots, a_n\}$ and $n \geq 2$,

$K = \{s = q_0, \ q_1, \ldots, q_n\}, \ F = K - \{q_0\}$, or $F = K$ and

$$\Delta = \bigcup_{i=1}^{n} \{(q_0, e, q_i)\} \cup \bigcup_{i,j=1}^{n} \{(q_i, a_j, q_i) : \ i \neq j\}$$

$i \neq j$ means that a_i is missing in the loop at state q_i

PROPERTIES

Equivalence of Two Definitions

Equivalence of Two Definitions

Book Definition (BD)

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

Class Definition (CD)

Δ is a **finite set** and

$$\Delta \subseteq K \times \Sigma^* \times K$$

Fact 1

Any **(BD)** automaton M is a **(CD)** automaton M

Proof

The **(BD)** of Δ is a particular case of the **(CD)** as

$$\Sigma \cup \{e\} \subseteq \Sigma^*$$

Equivalence of Two Definitions

Fact 2

Any **(CD)** automaton M can be transformed into an equivalent **(BD)** automaton M '

Proof

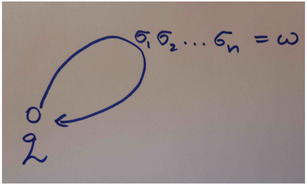
We use a " streching " technique

For any $w \neq e, \quad w \in \Sigma^*$ and **(CD)** transition $(q, w, q') \in \Delta$, we transform it into a **sequence** of **(BD)** transactions each reading only $\sigma \in \Sigma$ that will at the end read the whole word $w \in \Sigma^*$

We leave the transactions $(q, e, q') \in \Delta$ unchanged

Stretching Process

Consider $w = \sigma_1, \sigma_2, \ldots \sigma_n$ and a transaction $(q, w, q) \in \Delta$ as depicted on the diagram



We construct Δ' in M ' by **replacing** the transaction $(q, \sigma_1, \sigma_2, \ldots \sigma_n, q)$ by

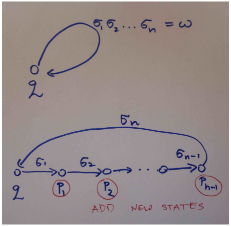
$$(q, \sigma_1, p_1),\ (p_1, \sigma_2, p_2),\ \ldots\ (p_{n-1}, \sigma_n, q)$$

and **adding** new states $p_1, p_2, \ldots p_{n-1}$ to the set K of M making at **this stage**

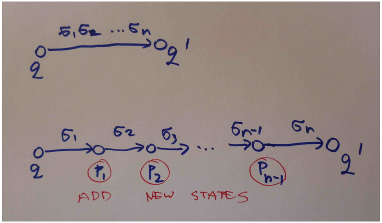
$$K' = K \cup \{p_1, p_2, \ldots p_{n-1}\}$$

Stretching Process

This transformation is depicted on the diagram below



We proceed in a similar way in a case of $w = \sigma_1, \sigma_2, \ldots \sigma_n$ and a transaction $(q, w, q') \in \Delta$



Equivalent M'

We proceed to do the "stretching" for all $(q, w, q') \in \Delta$ for $w \neq e$ and take as

$$K' = K \cup P$$

where $P = \{p : \ p$ added by stretching for all $(q, w, q') \in \Delta\}$

We take as

$$\Delta = \Delta^{\Sigma} \cup \{(q, \sigma_i, p) : \ p \in P, w = \sigma_1, \ \ldots \sigma_n, \ \ (q, w, q') \in \Delta\}$$

where

$$\Delta^{\Sigma} = \{(q, \sigma, q') \in \Delta : \ \ \sigma \in (\Sigma \cup \{e\}), \ \ q, q' \in K\}$$

Proof of Equivalency of DFA and NDFA

Equivalency of DFA and NDFA

Let's now go back now to the **Equivalency Statement** that consists of the following two equivalency theorems

Equivalency Theorem 1

For any DFA M, there is is a NDFA M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

This is already **proved**

Equivalency Theorem 2

For any NDFA M, there is a DFA M', such that $M \approx M'$, i.e. such that

$$L(M) = L(M')$$

This is to be proved

Equivalency Theorem

Our goal now is to prove the following

Equivalency Theorem 2

For any **nondeterministic** automaton

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

there is, i.e. we give an algorithm for its construction a **deterministic** automaton

$$M' = (K',\ \Sigma,\ \delta = \Delta',\ s',\ F')$$

such that

$$M \approx M'$$

i.e.

$$L(M) = L(M')$$

General Remark

We base the **proof** of the equivalency of DFA and NDFA automata on the **Book Definition** of NDFA

Let's now explore some **ideas** laying behind the main points of the **proof**

They are based on two **differences** between the DFA and NDF automata

We discuss now these **differences** and basic **ideas** how to overcome them, i.e. how to "make" a deterministic automaton out of a nonderetministic one

NDFA and DFA Differences

Difference 1

DFA **transition** function δ even if expressed as a **relation**

$$\delta \subseteq K \times \Sigma \times K$$

must be a function, while the NDFA transition relation Δ

$$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$$

may **not be** a function

NDFA and DFA Differences

Difference 2

DFA **transition** function δ **domain** is the set

$$K \times \Sigma$$

while NDFA **transition** relation Δ **domain** is the set

$$K \times \Sigma \cup \{e\}$$

Observe that the NDFA **transition** relation Δ may contain a configuration (q, e, q') that allows a nondeterministic automaton to **read** the empty word e, what is **not allowed** in the deterministic case

In order to **transform** a nondeterministic M into an equivalent deterministic M' we have to **eliminate** the both Differences 1 and 2

Example

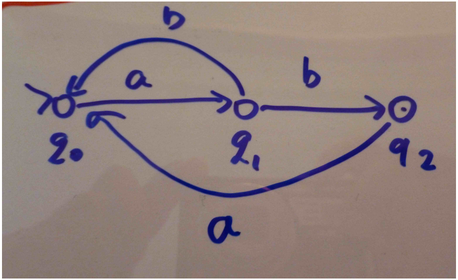
Let's look first at the following

Example

$$M = (\{q_0, q_1, q_2, q_3\},\ \Sigma = \{a, b\},\ \Delta,\ s = q_0,\ F = \{q_2\})$$

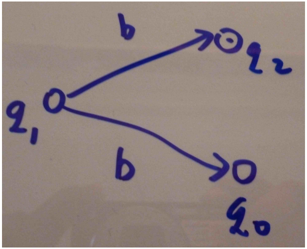
$$\Delta = \{(q_0, a, q_1), (q_1, b, q_0), (q_1, b, q_2), (q_2, a, q_0)\}$$

Diagram of M



Example

The **non-function** part of the diagram is



Question

How to transform it into a FUNCTION???

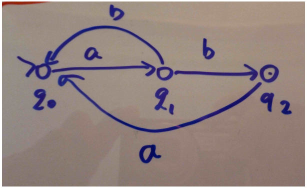
IDEA 1: make the states of M' as some SETS made out of states of M and put in this case

$$\delta(\{q_1\}, b) = \{q_0, q_2\}$$

IDEA ONE

IDEA 1: we make the states of M' as some SETS made out of states of M

We read other transformation from the **Diagram** of M



$\delta(\{q_0\}, a) = \{q_1\}, \quad \delta(\{q_2\}, a) = \{q_0\}$ and of course
$\delta(\{q_1\}, b) = \{q_0, q_2\}$

We make the state $\{q_0\}$ the **initial state** of M' as q_0 was the initial state of M and

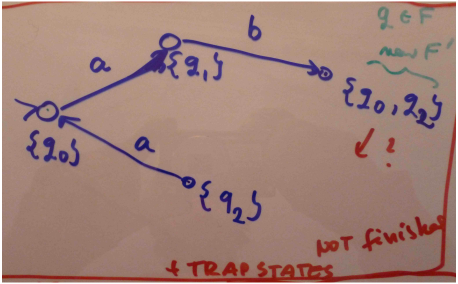
we make the states $\{q_0, q_2\}$ and $\{q_2\}$ **final states** of M' and as q_2 was a **final state** of M

Example

We have constructed a part of

$$M' = (K', \Sigma, \delta = \Delta', s', F')$$

The **Unfinished Diagram** is

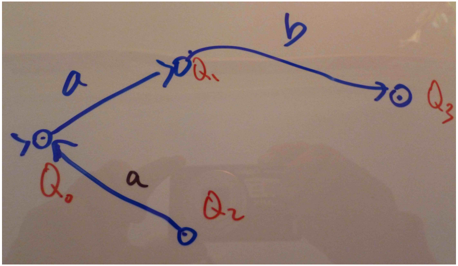


There will be many **trap states**

Example Revisited

In the case of our **Example** we had $K = \{q_0, q_1, q_2\}$
$K' = 2^K$ has 2^3 states

The portion of the **unfinished diagram** of M' is



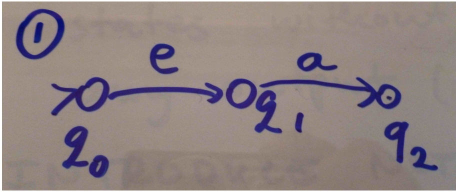
It is obvious that even the finished diagram will have A LOT of **trap states**

Difference 2 and Idea Two

Difference 2 and Idea Two - how to eliminate the e transitions

Example 1

Consider M1



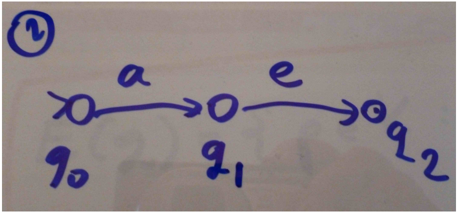
Observe that we can go from q_0 to q_1 reading only e, i.e. without reading any **input** symbol $\sigma \in \Sigma$

$$L(M1) = a$$

Examples

Example 2

Consider M2

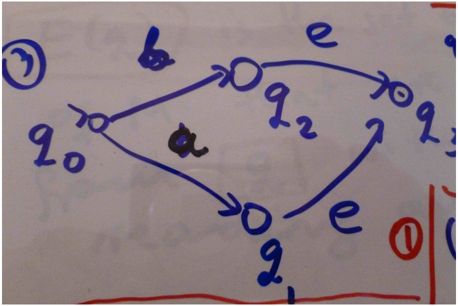


Observe that we can go from q_1 to q_2 reading only e, i.e. without reading any **input** symbol $\sigma \in \Sigma$

$$L(M2) = a$$

Example 3

Consider M3



Observe that we can go from q_2 to q_3 and from q_1 to q_3 without reading **any input**

$$L(M3) = a \cup b$$

Idea Two - Sets E(q)

The definition of the **transition function** δ of M' uses the following

Idea Two: a move of M' on reading an input symbol $\sigma \in \Sigma$ **imitates** a move of M on input symbol σ, possibly followed by **any** number of e-moves of M

To formalize this idea we need a special definition

Definition of E(q)

For any state $q \in K$, let E(q) be the set of all states in M they are **reachable** from state q without reading **any input**, i.e.

$$E(q) = \{p \in K : \ (q, e) \vdash_M{}^* (p, e)\}$$

Sets E(q)

Fact 1

For any state $q \in K$ we have that $q \in E(q)$

Proof

By definition

$$E(q) = \{p \in K : \ (q, e) \vdash_M{}^* (p, e)\}$$

and by the definition of reflexive, transitive closure $\vdash_M{}^*$ the trivial path (case n=1) always exists, hence

$$(q, e) \vdash_M{}^* (q, e)\}$$

what proves that $q \in E(q)$

Sets E(q)

Observe that by definitions of $\vdash_M{}^*$ and E(q) we have that

Fact 2

1. E(q) is a **closure** of the set $\{q\}$ under the relation

$$\{(p, r) : \text{ there is a transition } (p, e, r) \in \Delta\}$$

2. E(q) can be computed by the following

Algorithm

Initially set $E(q) := \{q\}$

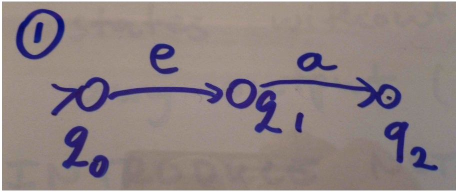
while there is $(p, e, r) \in \Delta$ with $p \in E(q)$ and $r \notin E(q)$

do: $E(q) := E(q) \cup \{r\}$

Example

We go back to the **Example 1**, i.e.

Consider M1



We evaluate

$$E(q_0) = \{q_0, q_1\}, \quad E(q_1) = \{q_1\}, \quad E(q_2) = \{q_2\}$$

Remember that always $q \in E(q)$

Definition of M'

Definition of M'

Given a **nondeterministic** automaton $M = (K,\ \Sigma,\ \Delta,\ s,\ F)$

we define the **deterministic** automaton M' equivalent to M as

$$M' = (K',\ \Sigma,\ \delta',\ s',\ F')$$

where

$$K' = 2^K, \quad s' = \{s\}$$

$$F' = \{Q \subseteq K : \quad Q \cap F \neq \emptyset\}$$

$\delta' : 2^K \times \Sigma \longrightarrow 2^K$ is such that and for each $Q \subseteq K$ and for each $\sigma \in \Sigma$

$$\delta'(Q, \sigma) = \bigcup \{E(p) : \quad p \in K \text{ and } (q, \sigma, p) \in \Delta \text{ for some } q \in Q\}$$

Definition of δ'

We re-write the definition of δ' in a a following form that is easier to use

$\delta' : 2^K \times \Sigma \longrightarrow 2^K$ is such that and for each $Q \subseteq K$ and for each $\sigma \in \Sigma$

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \ (q, \sigma, p) \in \Delta \text{ for some } q \in Q\}$$

or we write it in a more clear form as

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \exists_{q \in Q} (q, \sigma, p) \in \Delta\}$$

Construction of of M'

Given a **nondeterministic** automaton $M = (K, \Sigma, \Delta, s, F)$

Here are the **STAGES** to follow when constructing M'

STAGE 1

1. For all $q \in K$, **evaluate** E(q)

$$E(q) = \{p \in K : \ (q, e) \vdash_M{}^* (p, e)\}$$

2. **Evaluate** initial and final states: $s' = E(s)$ and

$$F' = \{Q \subseteq K : \ Q \cap F \neq \emptyset\}$$

STAGE 2

Evaluate $\delta'(Q, \sigma)$ for $\sigma \in \Sigma,\ Q \in 2^K$

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \ \exists_{q \in Q}\ (q, \sigma, p) \in \Delta\}$$

Evaluation of δ'

Observe that domain of δ' is $2^K \times \Sigma$ and can be very large

We will **evaluate** δ' only on states that are relevant to the **operation** of M' and making all other states **trap states**
We do so to **assure** that

$$M' \approx M$$

i.e. to be able to **prove** that

$$L(M) = L(M')$$

Having this in mind we adopt the following definition

Evaluation of δ'

Definition

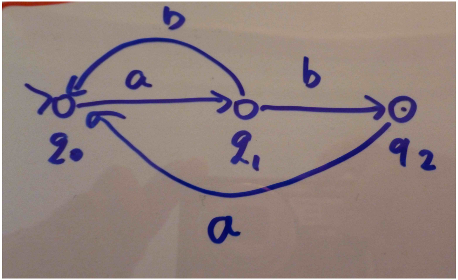
We say that a state $Q \in 2^K$ is **relevant** to the operation of M' and to the language L(M') if it can be **reached** from the **initial state** $s' = E(s)$ by reading some input string

Obviously, any state $Q \in 2^K$ that is **not reachable** from the **initial state** s' is **irrelevant** to the operation of M' and to the language L(M')

Construction of of M' Example

Example

Let M be defined by the following **diagram**



STAGE 1

1. For all $q \in K$, **evaluate** E(q)

M does not have e -transitions so we get

$E(q_0) = \{q_0\}, \quad E(q_1) = \{q_1\}, \quad E(q_2) = \{q_2\}$

2. Evaluate initial and some final states: $s' = E(q_0) = \{q_0\}$ and $\{q_2\} \in F'$

δ' Evaluation

STAGE 2

Here is a **General Procedure** for δ' evaluation

Evaluate $\delta'(Q, \sigma)$ only for **relevant** $Q \in 2^K$, i.e. follow the steps below

Step 1 Evaluate $\delta'(s', \sigma)$ for all $\sigma \in \Sigma$, i.e. all states **directly reachable** from s'

Step (n+1)

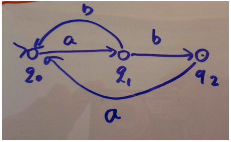
Evaluate δ' on all states that result from the **Step n**, i.e. on all states **already reachable** from s'

Remember

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \exists_{q \in Q}\ (q, \sigma, p) \in \Delta\}$$

Example STAGE 2

Diagram



STAGE 2

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \exists_{q \in Q} \, (q, \sigma, p) \in \Delta\}$$

Step 1 We evaluate $\delta'(\{q_0\}, a)$ and $\delta'(\{q_0\}, b)$

We look for the transitions from q_0

We have only one $(q_0, a, q_1) \in \Delta$ so we get

$\delta'(\{q_0\}, a) = E(q_1) = \{q_1\}$

There is no transition $(q_0, b, p) \in \Delta$ for any $p \in K$, so we get $\delta'(\{q_0\}, b) = E(p) = \emptyset$

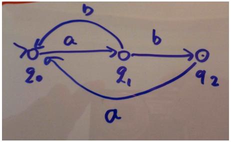
Example STAGE 2

By the **Step 1** we have that all states directly reachable from s' are $\{q_2\}$ and $\emptyset$

Step 2 Evaluate δ' on all states that result from the **Step 1**; i.e. on states $\{q_1\}$ and $\emptyset$

Obviously $\delta'(\emptyset, a) = \emptyset$ and $\delta'(\emptyset, b) = \emptyset$

To evaluate $\delta'(\{q_1\}, a)$, $\delta'(\{q_1\}, b)$ we first look at all transitions $(q_1, a, p) \in \Delta$ on the diagram

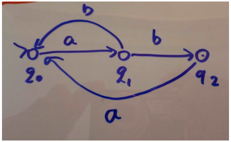


There is no transition $(q_1, a, p) \in \Delta$ for any $p \in K$, so

$$\delta'(\{q_1\}, a) = \emptyset \text{ and } \delta'(\emptyset, a) = \emptyset, \ \delta'(\emptyset, b) = \emptyset$$

Example STAGE 2

Step 2 To evaluate $\delta'(\{q_1\}, b)$ we now look at all transitions $(q_1, b, p) \in \Delta$ on the diagram



Here they are: (q_1, b, q_2), (q_1, b, q_0)

$\delta'(Q, \sigma) = \bigcup_{p \in K}\{E(p) : \exists_{q \in Q}\ (q, \sigma, p) \in \Delta\}$

$\delta'(\{q_1\}, b) = E(q_2) \cup E(q_0) = \{q_2\} \cup \{q_0\} = \{q_0, q_2\}$

We evaluated

$$\delta'(\{q_1\}, b) = \{q_0, q_2\}, \quad \delta'(\{q_1\}, a) = \emptyset$$

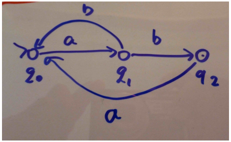
We also have that the state $\{q_0, q_2\} \in F'$

Example STAGE 2

Step 3 Evaluate δ' on all states that result from the **Step 2**; i.e. on states $\{q_0, q_2\}$, $\emptyset$

Obviously $\delta'(\emptyset, a) = \emptyset$ and $\delta'(\emptyset, b) = \emptyset$

To evaluate $\delta'(\{q_0, q_2\}, a)$ we look at all transitions (q_0, a, p) and (q_2, a, p) on the diagram



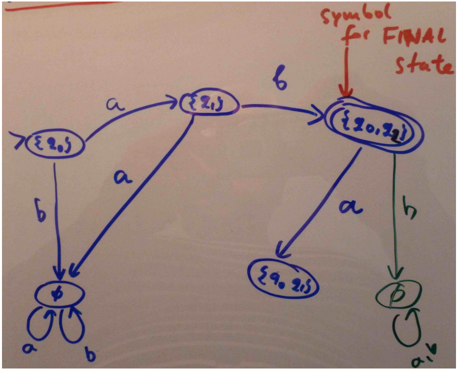
Here they are: (q_0, a, q_1), (q_2, a, q_0)

$$\delta'(\{q_0, q_2\}, a) = E(q_1) \cup E(q_0) = \{q_0, q_1\}$$

Similarly $\delta'(\{q_0, q_2\}, b) = \emptyset$

Diagram Steps 1 - 3

Here is the **Diagram** of M’ after finishing STAGE 1 and **Steps 1-3** of the STAGE 2

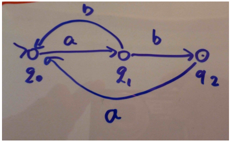


Example STAGE 2

Step 4 Evaluate δ' on all states that result from the **Step 3**; i.e. on states $\{q_0, q_1\}$, $\emptyset$

Obviously $\delta'(\emptyset, a) = \emptyset$ and $\delta'(\emptyset, b) = \emptyset$

To evaluate $\delta'(\{q_0, q_1\}, a)$ we look at all transitions (q_0, a, p) and (q_1, a, p) on the diagram



Here there is one (q_0, a, q_1), and **there is no** transition (q_1, a, p) for any $p \in K$, so

$$\delta'(\{q_0, q_1\}, a) = E(q_1) \cup \emptyset = \{q_1\}$$

Similarly

$$\delta'(\{q_0, q_1\}, b) = \{q_0, q_2\}$$

Example STAGE 2

Step 5 Evaluate δ' on all states that result from the **Step 4**; i.e. on states $\{q_1\}$ and $\{q_0, q_2\}$

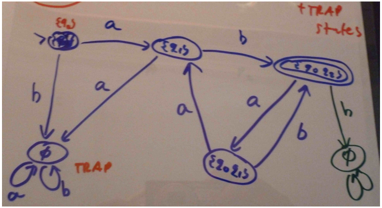
Observe that we have already evaluated $\delta'(\{q_1\}, \sigma)$ for all $\sigma \in \Sigma$ in **Step 2** and $\delta'(\{q_0, q_2\}, \sigma)$ in **Step 3**

The process of defining $\delta'(Q, \sigma)$ for **relevant** $Q \in 2^K$ is hence **terminated**

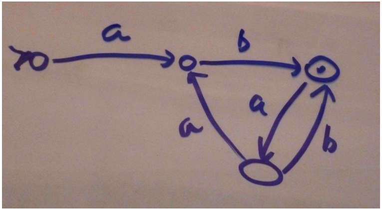
All other states are **trap states**

Diagram of of M’

Here is the **Diagram** of the **Relevant Part** of M’



and here is its **short pattern diagram** version

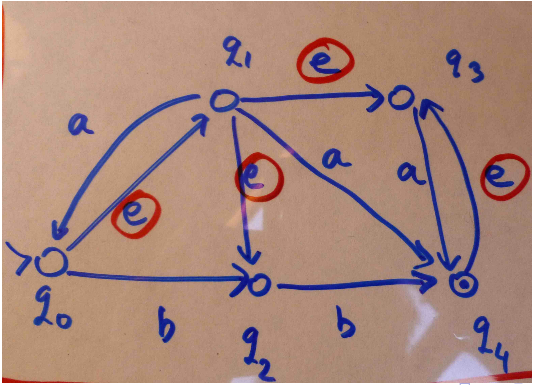


Book Example

Here is the nondeterministic M from book page 70

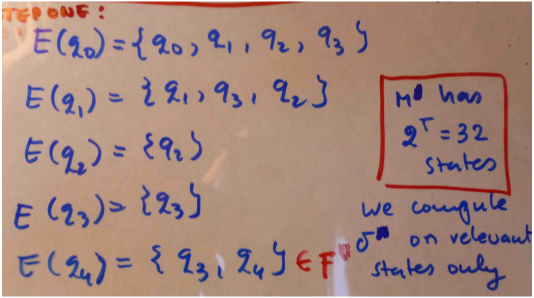
Exercise Read the example and re- write it as an exercise stage by stage as we did in class - it means follow the previous example

Diagram of M



Book Example

STAGE 1

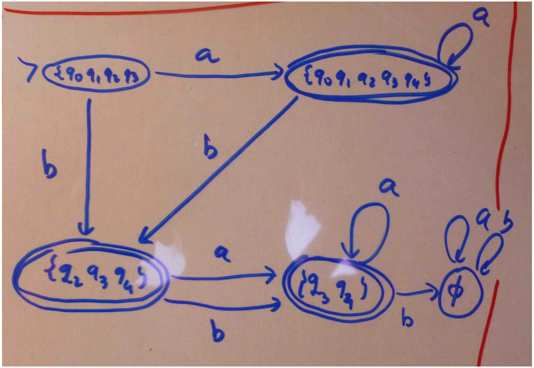


STAGE 2 evaluation are on page 72

Evaluate them independently of the book

Book Example

Diagram of M’



Book Example

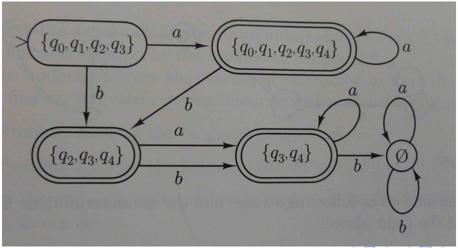
Some **book computations**

$$\delta'(\{q_0, q_1, q_2, q_3, q_4\}, a) = \{q_0, q_1, q_2, q_3, q_4\},$$
$$\delta'(\{q_0, q_1, q_2, q_3, q_4\}, b) = \{q_2, q_3, q_4\},$$
$$\delta'(\{q_2, q_3, q_4\}, a) = E(q_4) = \{q_3, q_4\},$$
$$\delta'(\{q_2, q_3, q_4\}, b) = E(q_4) = \{q_3, q_4\}.$$

$$\delta'(\{q_3, q_4\}, a) = E(q_4) = \{q_3, q_4\},$$
$$\delta'(\{q_3, q_4\}, b) = \emptyset,$$

$$\delta'(\emptyset, a) = \delta'(\emptyset, b) = \emptyset.$$

Book Diagram



NDFA and DFA Differences Revisited

Difference 1 Revisited

DFA transition function δ even if expressed as a relation

$\delta \subseteq K \times \Sigma \times K$

must be a function, while the NDFA transition relation Δ

$\Delta \subseteq K \times (\Sigma \cup \{e\}) \times K$

may **not be a function**

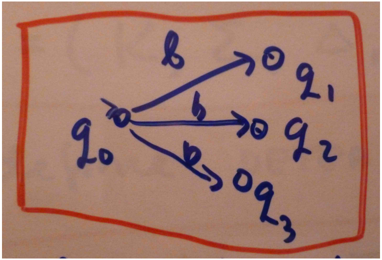
Difference 2 Revisited

DFA transition function δ **domain** is the set $K \times \Sigma$ while

It is obvious that the definition of δ' solves the **Difference 2**

Difference 1

Given a **non-function diagram** of M

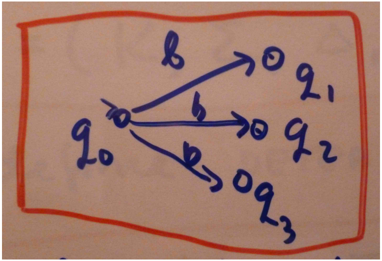


Proposed IDEA of f solving the **Difference 1** was to make the states of M' as some subsets of the set of states of M and put in this case

$$\delta'(\{q_0\}, b) = \{q_1, q_2, q_3\}$$

Exercise

Given the **diagram** of M



Exercise

Show that the definition of δ'

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) : \exists_{q \in Q}\ (q, \sigma, p) \in \Delta\}$$

does exactly what we have proposed, i.e show that

$$\delta'(\{q_0\}, b) = \{q_1, q_2, q_3\}$$

Proof of Equivalency Theorem

Equivalency Theorem

For any **nondeterministic** automaton

$$M = (K,\ \Sigma,\ \Delta,\ s,\ F)$$

there is (we have given an algorithm for its construction) a **deterministic** automaton

$$M' = (K',\ \Sigma,\ \delta = \Delta',\ s',\ F')$$

such that

$$M \approx M' \quad \text{i.e.} \quad L(M) = L(M')$$

Proof

M' is deterministic directly from the definition because the formula

$$\delta'(Q, \sigma) = \bigcup_{p \in K} \{E(p) :\ \exists_{q \in Q}\ (q, \sigma, p) \in \Delta\}$$

defines a function and is well defined for a all $Q \in 2^K$ and $\sigma \in \Sigma$.

Proof of Equivalency Theorem

We now claim that the following Lemma holds and we will prove equivalency $M \approx M'$ from the Lemma

Lemma

For any word $w \in \Sigma^*$ and any states $p, q \in K$

$$(q, w) \vdash_M^* (p, e) \quad \text{if and only if} \quad (E(q), w) \vdash_{M'}^* (P, e)$$

for some set P such that $p \in P$

We carry the **proof** of the **Lemma** by induction on the length $|w|$ of w

Base Step $|w| = 0$; this is possible only when t $w = e$ and we must show

$$(q, e) \vdash_M^* (p, e) \quad \text{if and only if} \quad (E(q), e) \vdash_{M'}^* (P, e)$$

for some P such that $p \in P$

Proof of Lemma

Base Step We must show that

$$(q, e) \vdash_M^* (p, e) \text{ if and only if } \exists_P (p \in P \cap (E(q), e) \vdash_{M'}^* (P, e)))$$

Observe that $(q, e) \vdash_M^* (p, e)$ just says that $p \in E(q)$ and the right side of statement holds for $P = E(q)$

Since M' is deterministic the statement $\exists_P (p \in P \cap (E(q), e) \vdash_{M'}^* (P, e)))$ is equivalent to saying that $P = E(q)$ and since $p \in P$ we get $p \in E(q)$ what is equivalent to the left side

This completes the proof of the basic step

Inductive step is similar and is given as in the book page 71

Proof of The Theorem

We have just proved that for any $w \in \Sigma^*$ and any states $p, q \in K$

$$(q, w) \vdash_M{}^* (p, e) \quad \text{if and only if} \quad (E(q), w) \vdash_{M'}{}^* (P, e)$$

for some set P such that $p \in P$

The **proof** of the **Equivalency Theorem** continues now as follows

Proof of The Theorem

We have to prove that L(M) = L(M')

Let' s take a word $w \in \Sigma^*$

We have (by definition of $L(M)$) that $w \in L(M)$

if and only if $(s, w) \vdash_M{}^* (f, e)$ for $f \in F$

if and only if $(E(s), w) \vdash_M{}^* (Q, e)$ for some Q such that $f \in Q$ (by the **Lemma**)

if and only if $(s', w) \vdash_M{}^* (Q, e)$ for some $Q \in F$ (by definition of M')

if and only if $w \in L(M')$

Hence L(M) = L(M')

This end the **proof** of the **Equivalency Theorem**

Finite Automata

We have proved that the class **(CD)** and book **(BD)** definitions of a nondeterministic automaton are **equivalent**

Hence by the **Equivalency Theorem** deterministic and ondeterministic automata defined by **any** of the both ways are **equivalent**

We will use now a name

FINITE AUTOMATA

when we talk about **deterministic** or **nondeterministic** automata