cse303
ELEMENTS OF THE THEORY OF COMPUTATION

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LECTURE 4a

REVIEW FOR CHAPTER 1

1. Some Short Questions
2. Some Homework Problems

CHAPTER 1

SHORT QUESTIONS

Short YES/NO Questions

Here are solutions to some short YES/NO Questions for material covered in CHAPTER 1

Solving Quizzes and Tests you have to write a short **solutions** and **circle** the answer

You will get **0 pts** if you only circle your answer without providing a **solution,** even if it is correct answer

Here are some questions

Short YES/NO Questions

Q1 $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset$

yes

We have that

$$\emptyset \in \{\emptyset\} \quad \text{and} \quad \emptyset \in \{\emptyset, \{\emptyset\}\}$$

This proves that

$$\emptyset \in \{\emptyset\} \cap \{\emptyset, \{\emptyset\}\}$$

Hence $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} \neq \emptyset$

Short YES/NO Questions

Q2 Some relations $R \subseteq A \times B$ are functions that map the set A into the set B

yes

Functions are special type of relations so some binary relations are functions (but not all relations are functions)

Q3 $2^{\emptyset} = \emptyset$

no

$\emptyset \subseteq \emptyset$ so $\emptyset \in 2^{\emptyset}$

Short YES/NO Questions

Q4 For any binary relation R on a set A, the inverse relation R^{-1} exists

yes

By definition of the inverse relation is

$$R^{-1} = \{(b, a) : \ (a, b) \in R\}$$

and such set always exists

Short YES/NO Questions

Q5 For any function $f : A \longrightarrow B$, the inverse function $f^{-1} : B \longrightarrow A$ exists

no

Inverse function to a function f exists if and only if f is $1-1$ and *onto*

Q6 A set $A = \{x \in N : \ x^2 + 5 < 0\}$ is countable

yes

$$A = \{x \in N : \ x^2 + 5 < 0\} = \emptyset$$

and any finite set is countable

Short YES/NO Questions

Q7 The set $A = \{n \in N : \ n^2 + 5 > 0\}$ is countable

yes

The definition says:

A set A is countable if and only if is finite

or is countably infinite

The set

$$A = \{n \in N : \ n^2 + 5 > 0\} = N$$

and N is countably infinite, hence A is countable

Short YES/NO Questions

Q8 The set $A = \{(\{n\}, n) \in 2^N \times N : \ 1 \leq n \leq n^2\}$ is infinitely countable

yes

First observe that

$$A = \{(\{n\}, n) \in 2^N \times N : \ 1 \leq n \leq n^2\} = C \times B$$

where the set B is

$$B = \{n \in N : \ 1 \leq n \leq n^2\}$$

and the set C is

$$C = \{\{n\} \in 2^N : 1 \leq n \leq n^2\}$$

Short YES/NO Questions

The condition $1 \leq n \leq n^2$ holds for all $n \in N - \{0\}$ hence the set

$$B = \{n \in N : \ 1 \leq n \leq n^2\}$$

is infinitely countable and so is the set

$$C = \{\{n\} \in 2^N : 1 \leq n \leq n^2\}$$

as the function $f(n) = \{n\}$ is $1 - 1$ and maps B onto C

The set

$$A = C \times B$$

is **infinitely countable** as it is the cartesian product of two infinitely countable sets

Short YES/NO Questions

Q9 Let $A = \{n \in N : n^2 + 1 \leq 15\}$

It is possible to define **8 alphabets** $\Sigma \subseteq A$

yes

The set

$$A = \{n \in N : n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}$$

so the set A has 4 elements and it has $2^4 = 16$ of all possible subsets and they are all finite, i.e we can define up to up to 16 alphabets $\Sigma \subseteq A$

So have can define for sure $8 < 16$ alphabets

Short YES/NO Questions

Q10 Let $\Sigma = \{n \in N : n^2 + 1 = 10\}$

There are **uncountably** many finite languages over Σ

no

Observe that

$$\Sigma = \{n \in N : n^2 + 1 = 10\} = \{3\}$$

and hence $|\Sigma^\star| = \aleph_0$

A finite language over Σ is by definition a finite subset of $\Sigma^\star$

We have a Theorem:

The set of all finite subsets of any countably infinite set is **countably infinite**

Short YES/NO Questions

Q11 For any languages $L_1,\ L_2,\ L$ over $\Sigma \neq \emptyset$ we have that

$$(L_1 \cup L_2) \cap L = (L_1 \cap L) \cup (L2 \cap L)$$

yes Languages are sets hence all laws of algebra of sets hold for them and this is one of the Distributivity laws

Q12 $L^* = \{w_1 w_2 \ldots w_n : \ \ w_i \in L,\ i = 1, 2, ..n,\ n \geq 1\}$

no

This is the definition of L^+ ; we must put $n \geq 0$ for L^*

Short YES/NO Questions

Q13 A **regular language** is a regular expression

no

A regular language is **represented** by a regular expression

More precisely, a regular language is **represented** by the function $\mathcal{L}$: Regular Expressions $\longrightarrow$ Regular Languages

such that the following holds

if α is any regular expression, then $\mathcal{L}(\alpha)$ is the language **represented** by α

Short YES/NO Questions

Q14 Let $\alpha = a(a \cup b)^*$

$$\mathcal{L}(\alpha) = \{w \in \{a,\ b\}^* :\ w \text{ ends with } a\}$$

no

We evaluate

$$\mathcal{L}(a(a \cup b)^*) = \{a\}(\{a\} \cup \{b\})^* = \{a\}\Sigma^*$$

and hence the property defining $\mathcal{L}(\alpha)$ is

$$\mathcal{L}(\alpha) = \{w \in \{a,\ b\}^* :\ w \text{ starts with } a\}$$

Short YES/NO Questions

Q15 For any language L over an alphabet Σ,

$$L^+ = L \cup L^*$$

no

Take L be any language such that $e \notin L$

We have that

$$e \notin L^+ \quad \text{but} \quad e \in L \cup L^*$$

This proves that

$$L^+ \neq L \cup L^*$$

CHAPTER 1

Some Homework Problems

Problem 1

Consider the following languages over $\Sigma = \{a, b\}$

$$L_1 = \{w \in \Sigma^\star : \exists u \in \Sigma\Sigma(w = uu^R u)\}$$

$$L_2 = \{w \in \Sigma^\star : ww = www\}$$

Part 1: Prove that L_1 is a finite set

Give example of 3 words $w \in L_1$

Solution

We evaluate first the set

$\Sigma\Sigma = \{a, b\}\{a, b\} = \{aa, bb, ab, ba\}$

$\Sigma\Sigma$ is a **finite set**, hence the set $B = \{uyu : u, y \in \Sigma\Sigma\}$

is also a **finite set** and by definition $L_1 \subseteq B$

This proves that L_1 must be a **finite set**

Problem 1

We evaluated that $\Sigma\Sigma = \{a, b\}\{a, b\} = \{aa, bb, ab, ba\}$

We defined $L_1 = \{w \in \Sigma^\star : \ \exists u \in \Sigma\Sigma(w = uu^R u)\}$

By evaluation we have that

$$L_1 = \{aaaaaa, abbaab, baabba, bbbbbb\}$$

Part 2: Give examples of 2 words over Σ such that $w \notin L_1$

Solution $a \notin L_1, \ \ bba \notin L_1$

There are countably infinitely many words that **are not** in L_1

Problem 1

Part 3 Consider now the following language

$$L_2 = \{w \in \{a, b\}^\star : \quad ww = www\}$$

Show that $L_2 \neq \emptyset$

Solution $e \in L_2$, as $ee = eee$

In fact, e is the only word with this property, hence

$$L_2 = \{e\}$$

Part 4 Show that the set $(\Sigma^\star - L_2)$ is infinite

Solution $\Sigma^\star$ is countably infinite, L_2 is finite, so (basic theorem) $(\Sigma^\star - L_2)$ is countably infinite

Any $w \in \Sigma^\star$, such that $w \neq e$ is in $(\Sigma^\star - L_2)$

Problem 2

Given expressions (written in a short hand notation)

$$\alpha_1 = \emptyset^\star \cup a^\star \cup b^\star \cup a \cup b \cup (a \cup b)^\star$$

$$\alpha_2 = (a \cup b)^\star b (a \cup b)^\star$$

Part 1 Re-write α_1 as a **simpler** expression representing the same language

List **properties** you used in your solution

Describe the language $L = \mathcal{L}(\alpha_1)$

Problem 2

Solution We first evaluate

$\mathcal{L}(\alpha_1) = \mathcal{L}(\emptyset^\star \cup a^\star \cup b^\star \cup a \cup b \cup (a \cup b)^\star)$

$= e \cup \{a\}^\star \cup \{b\}^\star \cup \{a\} \cup \{b\} \cup (\{a\} \cup \{b\})^\star = \Sigma^\star$

This is true because of the properties:

$$(\{a\} \cup \{b\})^\star = \{a, b\}^\star = \Sigma^\star \quad \text{and}$$

$\{a\} \subseteq \{a\}^\star, \ \{b\} \subseteq \{b\}^\star, \ \{a\}^\star \subseteq \Sigma^\star, \ \{b\}^\star \subseteq \Sigma^\star$

and we know that for any sets A, B, if $A \subseteq B$, then $A \cup B = B$

$\mathcal{L}(\alpha_1) = \Sigma^\star = (\{a\} \cup \{b\})^\star = \mathcal{L}((a \cup b)^\star)$

We hence simplify α_1 as follows

$$\alpha_1 = \emptyset^\star \cup a^\star \cup b^\star \cup a \cup b \cup (a \cup b)^\star = (a \cup b)^\star$$

Problem 3

Part 2 Given

$$\alpha_2 = (a \cup b)^\star b(a \cup b)^\star$$

Re-write α_2 as a **simpler** expression representing the same language

Describe the language $L = \mathcal{L}(\alpha_2)$

Solution α_2 can not be simplified, but we can use property $(\{a\} \cup \{b\})^\star = \Sigma^\star$ to describe informally the language determined by α_2 as

$$L = \mathcal{L}(\alpha_2) = \Sigma^\star b \Sigma^\star$$

Remember that informal description $\Sigma^\star b \Sigma^\star$ **is not** a regular expression - but just an **useful notation**

Problem 3

Let $\Sigma = \{a, b\}$ and a language $L \subseteq \Sigma^\star$ be defined as follows:

$$L = \{w \in \Sigma^\star : \ w \text{ contains no more then two } a\text{'s}\}$$

Write a regular expression α, such that $\mathcal{L}(\alpha) = L$. Use shorthand notation. **Explain** shortly your answer.

Solution

$$\alpha = b^\star \cup b^\star ab^\star \cup b^\star ab^\star ab^\star$$

Explanation

$b^\star$ contains 0 of a's (case n=0)

$b^\star ab^\star$ contains 1 occurrence of a (case n=1)

$b^\star ab^\star ab^\star$ contains 2 occurrence of a (case n=2)

Problem 4

Problem 4

Let $\Sigma = \{a, b\}$

The language $L \subseteq \Sigma^\star$ is defined as follows:

$L = \{w \in \Sigma^\star :$ the number of b 's in w is divisible by 4 $\}$

Write a regular expression α, such that $\mathcal{L}(\alpha) = L$

You can use **shorthand notation**. Explain shortly your answer

Solution

$\alpha = a^\star(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$

Observe that the regular expression $a^\star ba^\star ba^\star ba^\star ba^\star$ describes a string $w \in \Sigma^\star$ with **exactly four** b 's

Problem 4

The regular expression

$$(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$$

represents multiples of $w \in \Sigma^\star$ with **exactly four** b 's and hence words in which a number of b 's is **divisible by** 4

Observe that 0 is divisible by 4, so we need to add the case of 0 number of b 's, i.e. we need to include words $e, a, aa, aaa, , ...$

We do so by concatenating $(a^\star ba^\star ba^\star ba^\star ba^\star)^\star$ with a^* and get

$$L = a^\star(a^* ba^\star ba^\star ba^\star ba^\star)^\star$$

Problem 5

1. Let $A = \{(\{n, n+1\}, n) \in 2^N \times N : \; 1 \leq n \leq 3\}$

List all elements of A

Solution

1. By simple evaluation we get

$$A = \{(\{n, n+1\}, n) \in 2^N \times N : n = 1, 2, 3\}$$

$$= \{(\{1, 2\}, 1), (\{2, 3\}, 2), (\{3, 4\}, 3)\}$$

Problem 5

2. Let now $A = \{(\{n\}, n) \in 2^N \times N : \ 1 \leq n \leq n+1\}$

Prove that A is **infinitely countable**

Solution

Observe that the set A can be re-written as follows

$$A = \{(\{n\}, n) \in 2^N \times N : \ 1 \leq n \leq n+1\}$$

$$= \{(\{n\}, n) \in 2^N \times N : \ 1 \leq n\}$$

because $n \leq n+1$ **for all** $n \in N$

The set $B = \{\{n\} : n \in N\}$ has the same cardinality as N by the function $f(n) = \{n\}$

$A = B \times N$ is hence a Cartesian product of two infinitely countable sets, and as we have proved, an **infinitely countable** set

Problem 6

Let L be a language defines as follows

$$L = \{w \in \{a, b\}^* : \quad P(w)\}$$

for the property P(w) defined as follows

P(w): between any two a's in $w \in \{a, b\}^*$ there is an **even** number of **consecutive** b's

1. **Describe** a regular expression r such that $\mathcal{L}(r) = L$

   Remark that 0 is an even number, hence $a^* \in L$ and

   $r = b^* \cup b^*a^*b^* \cup b^*(a(bb)^*a)^*b^* = b^*a^*b^* \cup b^*(a(bb)^*a)^*b^*$

Problem 7

Let Σ be any alphabet, L_1, L_2 two languages over Σ such that $e \in L_1$ and $e \in L_2$

Show that

$$(L_1\Sigma^{\star}L_2)^{\star} = \Sigma^{\star}$$

Solution

By definition, $L_1 \subseteq \Sigma^{\star}, L_2 \subseteq \Sigma^{\star}$ and $\Sigma^{\star} \subseteq \Sigma^{\star}$

Hence

$$(L_1\Sigma^{\star}L_2) \subseteq \Sigma^{\star}$$

Problem 7

Now we use the following property:

Property

For any languages $L_1.L_2$,

if $L_1 \subseteq L_2$, then $L_1{}^* \subseteq L_2{}^*$

and obtain that $(L_1\Sigma^\star L_2)^\star \subseteq \Sigma^{\star *} = \Sigma^\star$, i.e. we proved that

$$(L_1\Sigma^\star L_2)^\star \subseteq \Sigma^\star$$

We have to show now that also

$$\Sigma^\star \subseteq (L_1\Sigma^\star L_2)^\star$$

Let $w \in \Sigma^\star$, we have that also $w \in (L_1\Sigma^\star L_2)^\star$ because $w = ewe$ and $e \in L_1$ and $e \in L_2$. We have hence **proved** that

$$(L_1\Sigma^\star L_2)^\star = \Sigma^\star$$

Problem 8

Problem 8

Let $\mathcal{L}$ be a function that associates with any regular expression α the regular language $L = \mathcal{L}(\alpha)$

1. Evaluate $L = \mathcal{L}(\alpha)$ for $\alpha = (a \cup b)^\star a$

Solution

$L = \mathcal{L}((a \cup b)^\star a) = \mathcal{L}((a \cup b)^\star)\mathcal{L}(a) = (\mathcal{L}(a \cup b))^\star\{a\} = (\mathcal{L}(a) \cup \mathcal{L}(b))^\star\{a\} = (\{a\} \cup \{b\})^\star\{a\} = \{a, b\}^\star\{a\}$

2 Describe a property that defines the language $L = \mathcal{L}((a \cup b)^\star a)$

Solution

$L = \{a, b\}^\star\{a\} = \Sigma^\star\{a\} = \{w \in \{a, b\}^\star :$ w ends with a $\}$